The mean value premium principle revisited*

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Abstract

In actuarial research, distortion-, mean value- and Haezendonck-Goovaerts risk measures are concepts that are usually treated separately. In this paper we indicate and characterize the relation between these different risk measures, as well as their relation to convex risk measures. First, we clarify the relation between the Haezendonck-Goovaerts risk measure and the distortion risk measure. Moreover, we explain the role provided for the distortion risk measures as an extension of the Tail Value-at-Risk (TVaR) and Conditional Tail Expectation (CTE).

While it is known that the mean value principle can be used to generate premium calculation principles, we show how they also allow to generate solvency calculation principles, by relating the mean value risk measures with the Haezendonck-Goovaerts risk measure. In a last step, a generalization is proposed, called the generalized Haezendonck-Goovaerts risk measure and the notions of comparability of this new risk measure are studied.

1 Introduction

A risk measure is a mapping from a class of random variables to the real numbers. A variety of risk measures is presented and discussed in the actuarial literature and the choice of the risk measure depends on the area in which it will be used. For example, risk measures used for premium calculations differ from the ones used for setting capital requirements and reserving. In the sequel, we will consider the random variable $X$, representing the loss an insurance company is facing. Its distribution function is denoted by $F_X$. The corresponding

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risk measure is denoted by $H(X)$. So $H(X)$ can represent the premium, charged by the insurance company to insure the risk $X$, or it can represent the solvency capital a company has to hold for being exposed to the risk $X$.

In this paper we will investigate three classes of risk measures: the distortion risk measures, the mean value principles and the Haezendonck-Goovaerts risk measures. We will show that a mean value principle can be used to define the Haezendonck-Goovaerts risk measure. Furthermore, we will use a distortion function to construct a more general version of the Haezendonck-Goovaerts risk measure, called the generalized Haezendonck-Goovaerts risk measure.

Throughout the paper, we use $\mathcal{B}$ to denote the class of all bounded risks and we only consider risks that belong to $\mathcal{B}$. Moreover, we will always silently assume that all the risks are non-negative.

## 2 Mean value risk measures

Mean value risk measures have been extensively studied. A beautiful characterization, based on the iterativity property, has been given by $?$; see also $?$. To derive this characterization, the following continuity condition was introduced.

**Definition 2.1 (Continuity condition)** Consider the random variable $X_{aq}$ which represents a risk:

$$\begin{align*}
\Pr(X_{aq} = a) &= q, \\
\Pr(X_{aq} = 0) &= 1 - q.
\end{align*}$$

(2.1)

For a fixed $a > 0$, the risk measure $H$ satisfies the continuity condition if, and only if, $H(X_{aq})$ is strictly increasing for $0 \leq q \leq 1$, with $H(X_{a0}) = 0$ and $H(X_{a1}) = a$.

A risk measure $H$ is said to satisfy the iterativity property if

$$H(X) = H\left( H(X | Y) \right),$$

where $X | Y$ is the conditional random variable $X$, given $Y$. Next, we recall the definition of the mean value principle.

**Definition 2.2** A risk measure $H$ is said to be generated by the mean value principle if there exists a strictly increasing function $v$ such that $v(H(X)) = E[v(X)]$.

**Theorem 2.1** A risk measure $H$ satisfying the continuity condition of Definition 2.1 is iterative if, and only if, it is generated by the mean value principle.

## 2.1 Comparing risk measures

**Definition 2.3 (Comparable mean value risk measures)** Two mean value principles $H$, with strictly increasing $v_1$ and $v_2$, are comparable if for all bounded risks $X$

$$H(X, v_1) \leq H(X, v_2),$$

or the reverse inequality, with $H(X, v) = v^{-1}(E[v(X)])$. 

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The following theorem is proven in ?.

**Theorem 2.2** Let $v_1$ and $v_2$ be two continuous and strictly increasing functions in $\mathbb{R}$. A necessary and sufficient condition for $H(X, v_1)$ and $H(X, v_2)$ to be comparable is that the function

$$f = v_2v_1^{-1}$$

satisfies

$$f (E [X]) \leq E [f (X)],$$

or the reverse inequality, for all bounded risks $X$. Hence, $f$ has to be a convex or a concave function.

### 3 Distortion risk measures

Consider a random variable $X$, belonging to the class $\mathcal{B}$. Its distribution function is denoted by $F_X$. An increasing distortion risk function $g : [0, 1] \rightarrow [0, 1]$, with $g(0) = 0$ and $g(1) = 1$, is used to calculate the distortion risk measure $\rho_g(X)$, for a (bounded and positive) loss $X$ as follows:

$$\rho_g(X) = \int_{0}^{\infty} g(1 - F_X(x)) \, dx. \quad (3.1)$$

We assume that $g$ is absolutely continuous. In this case, we can derive by means of a partial integration and a substitution the following equation:

$$\rho_g(X) = \int_{0}^{1} F_X^{-1}(y)g'(1 - y) \, dy. \quad (3.2)$$

The quantiles of a comonotonic sum $X^c + Y^c$ can be decomposed as follows:

$$F^{-1}_{X^c+Y^c}(p) = F^{-1}_{X^c}(p) + F^{-1}_{Y^c}(p), \quad p \in [0, 1].$$

Using expression (3.2), it is easy to show that distortion risk measures $\rho_g$ are additive for comonotonic risks $X^c$ and $Y^c$:

$$\rho_g(X^c + Y^c) = \rho_g(X^c) + \rho_g(Y^c).$$

### 4 Application of a mean value principle to generate distortion risk measures in the framework of capital requirements

It is known that positive homogeneous or translation invariant principles cannot be obtained when using a mean value principle to generate distortion risk measures. In the light of capital requirements, a good risk measure should focus more on the risk in the tails than on the total risk. Indeed, when setting an additional capital for potential heavy losses, one tries to avoid the situation where potential gains and potential (heavy) losses will balance each other.
out. For a bounded risk $X$, a risk capital $\rho(X)$ is determined using a transformed random variable $Z$, which only has probability mass in the right tail.

This section will show how to choose the variable $Z$, such that using a mean value principle to determine the risk $Z$ can be linked with a distortion risk measure and a Haezendonck-Goovaerts risk measure for the initial risk $X$.

4.1 Risk measures for capital requirements

Consider the random variable $Z$, which is defined as

$$Z = \frac{(F_X^{-1}(U) - t)}{\rho - t},$$

with $U$ uniformly distributed on the unit interval and $\rho, t \in \mathbb{R}^+$. We could apply the mean value risk measure for some valid function $v$ and obtain

$$v(H(Z)) = E[v(Z)].$$

For the moment, we take $v(x) = x$ for all $x$. If $H(Z)$ is forced to be equal to $1 - \alpha$, for some $\alpha \in (0, 1)$, the risk capital is denoted by $\rho_l(X, t)$. Here we use the subscript $l$ because a linear function $v$ is used. Given the available capital $t$, $\rho_l(X, t)$ can be derived from the following equation:

$$H(Z) = \int_0^1 \frac{(F_X^{-1}(u) - t)}{\rho_l(X, t) - t} du = 1 - \alpha,$$

from which it follows that the risk capital $\rho_l(X, t)$ is given by:

$$\rho_l(X, t) = t + \frac{1}{1 - \alpha} \int_t^{+\infty} (x - t) + dF_X(x).$$

Hence, $\rho_l(X, t) - t$ seems to be expressed as a distortion risk measure. However, the corresponding function $g(x) = \frac{x}{1 - \alpha}$ is not a distortion function because $g(1) = \frac{1}{1 - \alpha} > 1$.

We can "solve" this by taking $t = F_X^{-1}(\alpha)$ and defining the function $g$ as:

$$g(x) = \min \left\{ \frac{x}{1 - \alpha}, 1 \right\}.$$  \hspace{1cm} (4.3)

The function $g$ is now a valid distortion function and we find that

$$\rho_g(X, F_X^{-1}(\alpha)) = \rho_l(X, F_X^{-1}(\alpha)).$$ \hspace{1cm} (4.4)

Clearly, starting from a mean value principle applied on the transformed random variable $Z$, the risk capital $\rho_l(X, F_X^{-1}(\alpha))$ can be linked with the risk capital $\rho_g(X, F_X^{-1}(\alpha))$.  

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4.2 Haezendonck-Goovaerts risk measure

Consider a random variable $X$, satisfying $-\infty < \min \left[ X \right] \leq \max \left[ X \right] < +\infty$ and a function $\varphi$, which is strictly increasing and has $\varphi (0) = 0$, $\varphi (1) = 1$ and $\varphi (+\infty) = +\infty$. For any $t \in \mathbb{R}$ and $\rho > t$, we have from $\varphi$:

$$\Pr \left( X > \rho \right) = \Pr \left( X - t > \rho - t \right) \leq \mathbb{E} \left[ \varphi \left( \frac{(X - t)_+}{\rho - t} \right) \right].$$

It can be shown that the equation

$$\mathbb{E} \left[ \varphi \left( \frac{(X - t)_+}{\rho - t} \right) \right] = 1 - \alpha, \quad (4.5)$$

with $\alpha \in (0, 1)$, always has a solution, as long as $-\infty < t < \max \left[ X \right]$. This solution is denoted by $\rho_\varphi (X, t)$.

**Definition 4.1** Let $\varphi$ be a strictly increasing function with $\varphi (0) = 0$, $\varphi (1) = 1$, $\varphi (+\infty) = +\infty$ and let $\alpha \in (0, 1)$. The Haezendonck-Goovaerts risk measure is denoted by $\rho_\varphi (X)$ and equals:

$$\rho_\varphi (X) = \inf_{-\infty < t < \max \left[ X \right]} \rho_\varphi (X, t),$$

where $\rho_\varphi (X, t)$ is the solution of equation (4.5).

4.2.1 Properties

In case we calculate the Haezendonck-Goovaerts risk measure as follows, putting $t = F_X^{-1} (\beta)$:

$$1 - \alpha = \mathbb{E} \left[ \varphi \left( \frac{(F_X^{-1}(U) - F_X^{-1}(\beta))_+}{\rho_\varphi (X, t) - F_X^{-1}(\beta)} \right) \right], \quad (4.6)$$

the function $\rho_\varphi (X, t)$ is positive homogeneous and translation invariant. Indeed, we immediately see that:

$$\rho_\varphi (aX, t) = a \rho_\varphi (X, t), \quad \text{for } a > 0,$$

$$\rho_\varphi (a + X, t) = a + \rho_\varphi (X, t) \quad \text{for } a \in \mathbb{R}.$$ 

Therefore, the Haezendonck-Goovaerts risk measure is also homogeneous and translation invariant. In case $\varphi$ is concave, $\rho_\varphi$ is subadditive.

**Theorem 4.1** $^1$ Consider the function $\varphi$ which is assumed to be differentiable. The Haezendonck-Goovaerts risk measure $\rho_\varphi (X)$ is determined as the solution of the system of equations:

$$1 - \alpha = \int_t^{+\infty} \varphi \left( \frac{(x - t)_+}{\rho - t} \right) dF_X (x), \quad (4.7)$$

$$\rho = t + \frac{\int_t^{+\infty} \varphi' \left( \frac{(x - t)_+}{\rho - t} \right) (x - t)_+ dF_X (x)}{\int_t^{+\infty} \varphi' \left( \frac{(x - t)_+}{\rho - t} \right) dF_X (x)}. \quad (4.8)$$

$^1$This result has been presented by the authors at the meeting of the “Deutsche Gesellschaft für Versicherungs- und Finanzmathematik” (DGVFM), in April 2009.
Consider now the following question: given a distortion function \( g \) and the corresponding risk measure for a Bernoulli random variable/risk \( B_q \), can one determine the corresponding function \( \phi \) of the Haezendonck-Goovaerts risk measure to find back the distortion risk measure. The random variable \( B_q \) satisfies \( \Pr (B_q = 1) = 1 - \Pr (B_q = 0) = q \).

**Theorem 4.2** Consider a Bernoulli risk \( B_q \) and a function \( g(q) \) which is a distortion measure function such that \( g(q) \) is increasing for \( 0 < q < 1 - \alpha \) and \( g(q) = 1 \) for \( 1 - \alpha < q \leq 1 \). A sufficient condition for the existence of a function \( \phi \) for which the equality \( \rho_{\phi}(B_q) = \rho_{g}(B_q) \) holds, is that \( g(q) \) is convex for \( q \leq 1 - \alpha \).

### 4.3 The generalized Haezendonck-Goovaerts risk measure

**Definition 4.2 (Generalized Haezendonck-Goovaerts risk measure)** Let \( \varphi \) be a strictly increasing function with \( \varphi(0) = 0, \varphi(1) = 1, \varphi(+\infty) = +\infty \). Let \( \alpha \in (0, 1) \) and \( -\infty < t < F^{-1}_X(U) = \max(X) \). Given a distortion function \( h \), which is an increasing function, satisfying \( h(0) = 0 \) and \( h(1) = 1 \), the generalized Haezendonck-Goovaerts risk measure is denoted by \( \rho_{\varphi,h}(X, t) \) and is the solution of:

\[
1 - \alpha = E \left[ \varphi \left( \frac{(F^{-1}_X(U) - t)_+}{\rho_{\varphi,h}(X,t) - t} \right) h'(1 - U) \right], \tag{4.9}
\]

where \( U \sim \text{Uniform}(0,1) \).

If \( t \) is selected such that this generalized Haezendonck-Goovaerts risk measure is minimal, we find the so called optimal generalized Haezendonck-Goovaerts risk measure. The proof that (4.9) always has a solution proceeds along the same lines as in the proof given in ?. Theorems 4.3 and 4.4 derive necessary and sufficient conditions for two generalized Haezendonck-Goovaerts risk measures to be comparable.

**Theorem 4.3** Let \( \varphi \) be a continuous and strictly increasing function in \( \mathbb{R}^+ \) and \( h \) a valid distortion function. A necessary and sufficient condition such that \( \rho_{\varphi,h}(X, t) \) is comparable with and larger than \( \rho_{l,h}(X, t) \) (\( l \) means a linear Young function \( \varphi \)) is that \( \varphi \) is a convex function.

**Remark 4.1** It can be shown that a necessary and sufficient condition such that \( \rho_{\varphi,h}(X, t) \) is comparable with and smaller than \( \rho_{l,h}(X, t) \) is that \( \varphi \) is a concave function.

**Theorem 4.4** Let \( \varphi_1 \) and \( \varphi_2 \) be two continuous and strictly increasing functions on \( \mathbb{R} \) and \( h \) a valid distortion function. A necessary and sufficient condition that \( \rho_{\varphi_1,h}(X, t) \) is comparable with \( \rho_{\varphi_2,h}(X, t) \) is that

\[
f = \varphi_2 \varphi_1^{-1}
\]

should satisfy

\[
f(E[X]) \leq E[f(X)],
\]

or the reversed inequality for all \( X \in \mathcal{B} \).
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