Multidimensional Characteristic-based Upwind Finite Volume Method for Incompressible Flows on Unstructured Grids

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1 Introduction

This paper presents a new multidimensional characteristic based scheme (MCB) for numerical simulation of incompressible flows on unstructured grids in conjunction with artificial compressibility method. The original MCB scheme has been proposed by the authors already and this research is the extension of the mentioned scheme for unstructured grids. The significant difference between MCB and conventional characteristic-based scheme (CB) is the multidimensional nature of scheme which allows information propagates in any direction instead of only normal to the cell interface. In addition, the residual smoothing technique has been used for convergence accelerating. The accuracy and ability of proposed scheme has been studied by numerical tests for different Reynolds numbers and the results obtained using new scheme are in good agreement with the standard benchmark solution in the literature. The Navier-Stokes equations for two-dimensional incompressible flows modified by the artificial compressibility can be expressed as:

$$\int_{\Omega} \frac{\partial W}{\partial t} dV + \oint_{C} (F dS_x + G dS_y) = \frac{1}{Re} \oint_{C} (R dS_x + S dS_y)$$

(1)

where

$$W = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \quad F = \begin{bmatrix} \beta u \\ u^2 + p \\ uv \end{bmatrix}, \quad G = \begin{bmatrix} \beta v \\ vu \\ v^2 + p \end{bmatrix},$$

$$R = \begin{bmatrix} \frac{\partial u}{\partial x} \\ 0 \\ \frac{\partial v}{\partial x} \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

(2)
Here \( \mathbf{W} \) is the vector of primitive variables, \( \mathbf{F}, \mathbf{G} \) and \( \mathbf{R}, \mathbf{S} \) are convective and viscous flux vectors, respectively. The artificial compressibility parameter and Reynolds number are shown by \( \beta \) and \( \text{Re} \) respectively. The discretized form of Eqs. \((2)\) reads:

\[
A_i \frac{\partial \mathbf{W}_i}{\partial t} + \sum_{j=1}^{m} \mathbf{F}_j(\Delta S_x)_j + \sum_{j=1}^{m} \mathbf{G}_j(\Delta S_y)_j = \frac{1}{\text{Re}} \left[ \sum_{j=1}^{m} \mathbf{R}_j(\Delta S_x)_j + \sum_{j=1}^{m} \mathbf{S}_j(\Delta S_y)_j \right]
\]

where \( A_i \) is the cell area.

2 Solution algorithm

2.1 Two-dimensional characteristic structure for artificial compressibility equations

To derive the characteristic relations of incompressible flows, their corresponding "Euler equations" are considered \([1]\). These equations modified by artificial compressibility for deriving two-dimensional characteristic structures are:

\[
\begin{align*}
\frac{\partial p}{\partial t} + \beta \frac{\partial u}{\partial x} + \beta \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + u \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= 0
\end{align*}
\]

(4)

To obtain characteristic structure of equations, a characteristic surface in the form of \( f(x, y, t) = 0 \) is assumed. Using the kinematics relations for relating the partial derivatives to exact derivatives corresponding to the assumed surface, one gets the following system of equations \([2, 3]\):

\[
\begin{bmatrix}
\frac{\partial f}{\partial t} \\
\beta \\
f_x \\
f_y
\end{bmatrix}
\begin{bmatrix}
f_x \\
f_y \\
\psi \\
\psi
\end{bmatrix}
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(5)

where the subscripts stand for the partial differentiation and \( \psi \) is defined as:

\[
\psi = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y}
\]

(6)

For compatibility requirements of Eqs. \((5)\), the determinant of coefficient matrix is set to zero, hence:

\[
\psi = 0, \quad \psi = \frac{\beta}{f_t} \left( f_x^2 + f_y^2 \right)
\]

(7)
We assumed the pseudo-velocity vector \( \mathbf{V} = (u, v, 1) \) and normal vector to characteristic surface \( \mathbf{n} = (\cos(\varphi), \sin(\varphi), n_t) \) alike compressible Euler equations \([4]\), in which \( \varphi \) shows the wave direction. Expressing Eq. (7) in terms of vectors \( \mathbf{V} \) and \( \mathbf{n} \), two types of characteristic surface corresponding to following relations are obtained:

\[
\mathbf{V} \cdot \mathbf{n} = 0, \quad \mathbf{V} \cdot \mathbf{n} = \frac{\beta}{n_t}
\]

where \( n_t = \frac{f_t}{\sqrt{f_x^2 + f_y^2}} \) denotes the t-component of normal vector. By some mathematical operations, \( n_t \) takes the following forms:

\[
n_t = -\frac{(u \cos(\varphi) + v \sin(\varphi)) \pm \sqrt{(u \cos(\varphi) + v \sin(\varphi))^2 + 4\beta^2}}{2} = n_1, n_2
\]

Regarding the dual roots of second relation at Eqs. (8) as a function of \( n_t \), similar to compressible Euler equations, dual characteristic surfaces would exist. With straightforward mathematical operations it can be proven that the roots have always different signs. This depicts the growth of zones of influence and dependence around the pseudo streamlines. It can be shown that the characteristic paths equations are given by:

\[
\frac{dx}{dt} = u - \frac{\beta}{n_t} \cos(\varphi), \quad \frac{dy}{dt} = v - \frac{\beta}{n_t} \sin(\varphi)
\]

where \( \varphi \) is wave angle. As it is seen in Fig. (1), for any angle in the range \( 0 \leq \varphi \leq 2\pi \) there exist two bicharacteristics. The compatibility relations corresponding to characteristic paths (Eq. [10]) are obtained as:

\[
\frac{\beta}{n_t} du + \cos(\varphi)dp = 0, \quad \frac{\beta}{n_t} dv + \sin(\varphi)dp = 0
\]

Eqs. (11) are valid for both \( n_t = n_1, n_2 \) which shows the governing compatibility relations along bicharacteristics. For more details see \([5, 6]\).

2.2 Numerical scheme for evaluation of convective fluxes on unstructured grids

Using compatibility Eqs. (11), a new multidimensional characteristic based upwind scheme has been presented here. As seen in Fig. (2), four characteristic paths corresponding to four wave angles \( \varphi_1, \varphi_2, \varphi_3, \varphi_4 \) have been selected and the compatibility Eqs. (11) along them are used for evaluating convective flux between two cells “R” and “L”. When angle “\( \varphi_1 \)” (Fig. 2) is smaller than \( \pi/4 \), discretization is done by using the first relation of Eqs. (11) for “\( \varphi_3, \varphi_4 \)” and the second one for “\( \varphi_1, \varphi_2 \)” and when “\( \varphi_1 \)” is bigger than \( \pi/4 \) vice versa. For example, in the case of
When $\varphi_1 < \pi/4$, the discretized equations are in the following form:

\begin{align}
  p^* - p_1 + A(u^* - u_1) &= 0, \\
  p^* - p_2 + B(u^* - u_2) &= 0, \\
  p^* - p_3 + C(v^* - v_3) &= 0, \\
  p^* - p_4 + D(v^* - v_4) &= 0
\end{align}

Where $p^*$, $u^*$ and $v^*$ are the values at cell interface and $A$, $B$, $C$, $D$ can be obtained from Eqs. (9) and (11) as following:

\begin{align}
  \gamma_1 &= u_1 \cos(\varphi_1) + v_1 \sin(\varphi_1), \\
  \gamma_2 &= u_2 \cos(\varphi_2) + v_2 \sin(\varphi_2), \\
  A &= \frac{2\beta}{\cos(\varphi_1) \left[-\gamma_1 + \sqrt{\gamma_1^2 + 4\beta}\right]}, \\
  B &= \frac{2\beta}{\sin(\varphi_1) \left[-\gamma_1 + \sqrt{\gamma_1^2 + 4\beta}\right]}, \\
  C &= \frac{2\beta}{\cos(\varphi_2) \left[-\gamma_2 + \sqrt{\gamma_2^2 + 4\beta}\right]}, \\
  D &= \frac{2\beta}{\sin(\varphi_2) \left[-\gamma_2 + \sqrt{\gamma_2^2 + 4\beta}\right]}. \tag{13}
\end{align}

$p^*$, $u^*$ and $v^*$ are calculated by Eq. (13) using flow properties at points 1, 2, 3 and 4 in the previous time level. Then they are used to determine convective fluxes at the cell interface. The value of $u^*$ is determined from the first and second equations of Eq. (13) and $v^*$ is determined from third and fourth ones. The final value of $p^*$ is assumed to be the arithmetic average of values obtained from two sets of equations (first-second and third-fourth at Eq. (13)). Flow properties at points 1, 2 are set to neighborhood cell values and for points 3, 4 interpolated from two cells containing assumed face (cells "L, R" in Fig. 2). By using the flow values at points 3 and 4 in order to evaluating interface values at face $j$, we take into account the real two-dimensional nature of flow and don’t assume any one-dimensional assumptions.

### 3 Results

To compare the accuracy of newly proposed MCB scheme on unstructured grids, a series of tests were conducted at different Reynolds numbers for the lid driven cavity flow that two numbers of them are reported in extended abstract. An example of computational unstructured grid that is used for finite-volume MCB flow solver is shown in Fig. 3. Fig. 4 shows streamlines of the fine grid solutions for a 100 and 10,000 Reynolds numbers using MCB scheme. Figs. 5 and 6 present the $u$-velocity profile along a vertical line and $v$-velocity profile along a horizontal line passing through the cavity center. These profiles are in good agreement with the well known benchmark results of Ghia et al. [7] that are shown by symbols in the figures.
Figure 1: Characteristic structure for incompressible flow defined by artificial compressibility equations.

Figure 2: MCB stencil for evaluating convective fluxes.

Figure 3: Numerical grid used for a test case.
Figure 4: Streamlines of primary and secondary vortices, $Re = 100$ and 10,000.

Figure 5: Comparison of the predicted mid-plane velocity profiles for $u$ at, $Re = 100$ and 10,000.

Figure 6: Comparison of the predicted mid-plane velocity profiles for $v$ at, $Re = 100$ and 10,000.
References


