Bond and Swap pricing with interest rates driven by fBm

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1 Introduction.

Recently it has been shown that interest rate data exhibits long memory behaviour \[2\]. Fractional Brownian motion (fBm) \( B_t^H \) \((t \geq 0)\) has been introduced by Kolmogorov to study the turbulence in an incompressible fluid flow \[6\]. Long range dependence and self-similarity properties of fBm announced itself in the Nile river level studies of Hurst \[5\]. The parameter \( H \in (0, 1) \) which characterizes fBm is called Hurst parameter. Mandelbrot and van Ness studied properties of the fBm in \[7\]. In this work we assume that short term interest rate \( r_t \) is governed by the SDE driven by fBm of the form

\[
dr_t = u(r_t, t)dt + w(r_t, t)dB_t^H, \quad t \geq 0.
\]

where \( dB_t^H \) \((H \in (1/2, 1))\) is the infinitesimal increment of fBm \( B_t^H(t) \). Furthermore, we assume that \( u(r, t) \) and \( w(r, t) \) are sufficiently smooth functions of their arguments (see \[3\] for the existence and the uniqueness of the solutions). Solutions to \( (1) \) are stochastic processes on the probability space \((P, \Omega, U)\). We derive the bond pricing partial differential equation (BPE) for this problem. We find the special solutions which subsumes the fBm version of Vasicek and CIR models. Finally we obtain the solutions in the case of interest rate swap problem.

A zero-mean Gaussian process \( B_t^H \) with

\[
E[B_t^H B_s^H] = \frac{1}{2}[t^{2H} + s^{2H} - |t - s |^{2H}], \quad H \in (0, 1),
\]

is called fBm. Here \( E[\cdot] \) is expectation operator with respect to probability measure \( P[7] \). Some properties of fBm \( B_t^H \) is in order \[7\]

\[
E[(B_t^H)^2] = t^{2H}, \quad B_0^H = 0.
\]
ii) Although fBm is nonstationary this property implies that $W_t^H$ has stationary increments

$$E[(B_t^H - B_s^H)^2] = |t - s|^{2H}.$$ 

iii) Correlation function has the form

$$C(s) = E[(B_{t+1}^H - B_t^H)(B_{t+s+1}^H - B_{t+s}^H)] = \frac{1}{2}[|s+1|^{2H} + |s-1|^{2H} - 2|s|^{2H}] .$$

In Fig. 1 $C(s)$ has been plotted for various values of $H$. iv) Although $B_t^H$ is continuos in the sense of Kolmogorov it is nowhere differentiable. fBm is a self-similar process i.e., for any constant $a > 0$ the process $a^{-H}B_{at}^H$ and $B_t^H$ have the same distributio. Standard Brownian motion becomes a special case of the fBm with the Hurst parameter $H = 1/2$. This property alone arouse great interest in the literature. In contrast with the standard Brownian motion, increments of $B_t^H (t \geq 0)$ are no longer statistically independent for nonoverlapping intervals of $t$. The correlation between the increments of $B_t^H (t \geq 0)$ for nonoverlapping intervals become positive for $H \in (1/2, 1)$ as it is seen from Fig. 1. It shows long-range dependence i.e. it becomes persistent. This property motivates us to consider fBm to study long memory behaviour of interest rates.

An fBm $B_t^H (t \geq 0)$ is neither a semi-martingale nor a Markov process. Therefore, a new stochastic calculus is needed for its treatment. We rely on a Itô formula for $H \in (1/2, 1)$ and it has been given in [1].
2 Bond Pricing Equation with fBm

An interest rate derivative product which pays an amount \( Z \) at maturity \( T \) is called bond. Bond prices depend on two variables; such as interest rate and time. We denote bond price as \( V(r,t) \). The value of the bond price at maturity time is a constant \( Z \). So the terminal condition is \( V(r,T) = Z \). Now we set up a portfolio containing two bonds with maturity times \( T_1 \) and \( T_2 \), we denote the portfolio \( \Pi = V_1(r,t) - \Delta V_2(r,t) \). Change in the value of portfolio in “\( dt \)” is \( d\Pi = \frac{d\Pi}{dt} dt \). Hence

\[ d\Pi = dV_1(r,t) - \Delta dV_2(r,t). \]

By using Itô formula \([1]\) and no arbitrage possibility condition (conservation of money!) we obtain the BPE

\[ \frac{\partial V}{\partial t} + w \frac{\partial^2 V}{\partial r^2} \int_0^t w(s,r) \phi(s,t) ds + (u(r,t) - w(r,t) \lambda(r,t)) \frac{\partial V}{\partial r} - rV - K(r,t) = 0. \]

(2)

where \( \phi(s,t) = H(2H-1)|s-t|^{2H-1} \), \( \lambda(r,t) \) is the market price of risk and \( K(r,t) \) is the coupon paid by bond.

We next seek for a solution of the form

\[ V(r,t) = Z \exp \left( A(t,T) - rB(t,T) \right). \]

(3)

We let \( A(T, T) = 0 \) and \( B(T, T) = 0 \) in order to satisfy the terminal condition. We consider the case \( w(t,r_t) = w(r_t) \) to obtain the form of the stochastic interest rate as

\[ dr_t = (-\gamma r_t + \eta + \lambda w) dt + \sqrt{\alpha r_t - \beta dB_t^H}. \]

(4)

BPE \([2]\) admits the bond price

\[ V(r, t) = Z \exp \left( \frac{1}{\tau} \left( 2\beta \left( \frac{\delta \tau}{(\gamma + \delta)(\gamma + \delta - 2)} \right) \right) \left( \frac{\delta \tau}{(\gamma + \delta)(\gamma + \delta - 2)} \right) - r t \right) - rT \right) \right) \left( \frac{1}{2\delta} \left( \frac{1}{(\gamma + \delta) \left( e^{(\delta)(T-t)} - 1 \right) + 2\delta} \right) \right) \]

\[ + \ln \left( \frac{2\delta e^{(T-t)} \gamma + \delta}{\gamma + \delta} \right) \left( \frac{1}{(\gamma + \delta) \left( e^{(\delta)(T-t)} - 1 \right) + 2\delta} \right) \left( \frac{2\eta}{\alpha} \right) \right) \]

when the stochastic interest rate is modelled by generalized CIR \([4]\) model given in \([4]\). Here \( \tau = T - t \) and \( \alpha, \beta, \gamma, \delta = \sqrt{\gamma^2 + 22} = \delta \) and \( \eta \) are constants.

We also derive a PDE for interest rate swap. Solution to this problem is presented. Furthermore, we also discuss long memory model for Turkish interest rate data.
**References**


