Stochastic Credit Default Swap Pricing

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Extended Abstract

After US Credit Crunch (2008 Credit Crisis) market participants have became aware of importance of credit risk, its measurement and management practices once again. Most of the leading credit institutions and academicians have started to discuss the assumptions of credit risk models and the ability of models to evaluate complex structured products. It was the first time that the credit derivative instruments roll up with their huge size and measurement problem. Being the most sizeable and frequently used credit derivative Credit Default Swaps (CDS) valuation is a crucial issue to assess credit risk of financial, non-financial institutions and sovereigns.

Our aim is to examine the structural and intensity based credit risk models and measure the CDS prices by using the function for default obtained from the Merton model and the Cox-Ross constant barrier model.

1 Structural Models

Structural models (firm value models) evaluate a firm default, considering its financial asset values. The given asset values of firm \( \{V_t : 0 \leq t \leq T\} \) the default occur when \( V_t \) goes under some constant boundary \( L \) that prescribed or constructed as stochastic process \( \{L_t : 0 \leq t \leq T\} \) before or at maturity \( T \). These models are based on option pricing methodology just taking firm value as asset price and value of firm liabilities as strike price of option to evaluate the firm equity prices.

1.1 Merton Model

Robert C. Merton was the first to publish a paper explaining the firm value by modifying Black-Scholes option pricing formula. Thus, Merton model is the base of structural models. All other structural models are just derivatives of that model.

Robert C Merton models a firm value \( V \), through the time can be described by a diffusion-type stochastic process with stochastic differential equation.

\[ dV_t = \mu V_t dt + \sigma V_t dW_t \] (1)
\[ V_t = V_0 \exp \left\{ (\mu - \frac{1}{2} \sigma^2) + \sigma W_t \right\} \]  \hspace{1cm} (2)

where \( V_0 > 0 \). Therefore,

\[ \log \left( \frac{V_t}{V_0} \right) \sim \Phi \left\{ (\mu - \frac{1}{2} \sigma^2)t, \sigma^2t \right\} \]  \hspace{1cm} (3)

\( V_t \): firm value at time \( t \) where \( t \in [0, T] \)

\( Z^T_t \): value of a single zero coupon bond at time \( t \) with maturity \( T \) and

\( E_T \): value of equity at time \( T \).

Merton take equity price of a firm as option premium written on asset of firm with strike price equal to face value of liabilities, i.e., the face value of zero coupon bond. Therefore,

\[ V_t = Z^T_t + E_t \]  \hspace{1cm} (4)

due to the fact that the payoff at maturity \( T \) is equal to \( E_T \)

\[ E_T = \max[V_T - L, 0] = (V_T - L)^+ = \begin{cases} (V_T - L) & \text{if } V_T \geq L \\ 0 & \text{if } V_T < L \end{cases} \]

\[ E_T = x \Phi(d_1) - \exp \left( -r(T-t) \right) L \Phi(d_2) \]  \hspace{1cm} (5)

where

\[ d_1 = \frac{\log(x/L) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = d_1 - \sigma \sqrt{T-t} \]

According to Merton Model the default can only be happen at maturity \( T \)

\[ P_S(T/F_t) = \Phi(d_2) \]  \hspace{1cm} (6)

for

\[ d_2 = \frac{\log(V_t/L) + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} \]

where \( \Phi(x) = \int_{-\infty}^{x} \exp(-\frac{z^2}{2}) \, dz \) is the cumulative standard normal distribution.
1.2 The Black-Cox Model With Constant Barrier

As we mentioned before all other structural models are just extensions of Merton model. This model was developed by Black and Cox in 1976 and they assume that there will be default if asset values of firm goes under the constant barrier $L$. As oppose to Merton, this model allows us to assume that the firm has more than one liability having maturities less than $T$. In Merton model $L$ was considered as face value of liability, face value of single zero coupon bond, but here it can be any level that makes sense.

Since the Black-Cox model assumes that default can take place at any time before time $T$, which implies that when asset values $\{V_t: 0 \leq t \leq T\}$ hits the threshold $L, L \leq V_0$. This value is also called the first passage time.

Consequently, defining the stochastic differential equation of firm value $V_t$ as geometric Brownian motion under risk natural probability

$$dV_t = V_t \left( \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t \right)$$

and $\tau$ as default time $\tau = \inf\{t > 0; V_t \leq L\}$, then probability of survive can be found as

$$P_S(\tau/F_t) = P(\tau > T/\tau > t) = \Phi(d_3) - \left( \frac{L}{V_t} \right)^{2r - 1} \Phi(d_4) \quad (7)$$

where

$$d_3 = \frac{\log(V_t/L) + (r - \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}}$$

and

$$d_4 = d_3 - \sigma \sqrt{T - t}$$

2 CDS Pricing Using Structural Models

The Credit Default Swap (CDS) is a bilateral contract that signed between credit protection buyer and seller. The CDS premium is the price that periodically paid to the credit protection seller buy protection buyer for compensation of underlying credit risk exposed assets. Therefore the price or premium of CDS can be found by equalizing the present value of premium will paid and present value of probable future price of compensation (contingent claim) that subject to default or credit event.

Let $D(0,t_i)$ be the discounting factor for time $t_i$ and $\Delta t_i$ the time difference between two consecutive payments $t_i$ and $t_{i-1}$, then present value of CDS premium paid by protection buyer is

$$PV_P = c N \sum_{i=1}^{n} D(0,t_i) P_S(t_i) \Delta t_i \quad (8)$$
Hence, the present value of contingent payment paid by protection seller turns to be

\[ PV_L = (1 - R) \sum_{i=1}^{n} D(0, t_i) \mathbb{P}_S(t_{i-1}) - \mathbb{P}_S(t_i) \]  

(9)

Since \( PV_F = PV_L \), it follows that

\[ cN \sum_{i=1}^{n} D(0, t_i) \mathbb{P}_S(t_i) \Delta t_i = (1 - R) \sum_{i=1}^{n} D(0, t_i)[\mathbb{P}_S(t_{i-1}) - \mathbb{P}_S(t_i)] \]

and hence, we have

\[ c = \frac{(1 - R) \sum_{i=1}^{n} D(0, t_i)[\mathbb{P}_S(t_{i-1}) - \mathbb{P}_S(t_i)]}{\sum_{i=1}^{n} D(0, t_i) \mathbb{P}_S(t_i) \Delta t_i} \]  

(10)

in the discrete time setting. On the other hand, in continuous time this turns to be

\[ c = \frac{(1 - R)[- \int_{0}^{T} D(0, s) \mathbb{P}_S(s)]}{\int_{0}^{T} D(0, s) \mathbb{P}_S(s) ds} \]  

(11)

Consequently, we get the following expression by using the probability of survive in (6) that we estimate from Merton model:

\[ c = \frac{(1 - R) \left[ - \int_{0}^{T} D(0, s) \phi(d_2) \frac{\log \left( \frac{L}{V_s} \right) + (r - \frac{1}{2} \sigma^2)(s - t)}{2\sigma(s - t)^{\frac{3}{2}}} ds \right]}{\int_{0}^{T} \int_{-\infty}^{d_2} D(0, s) \phi(d_2) ds} \]  

(12)

where \( \phi(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \). Furthermore, CDS premium is also estimated from Cox-Ross model by same methodology by using equation (7).

References


