Extended Abstract

1 Introduction.

A Constant Proportion Debt Obligation (CPDO) is a structured credit product which is designed for offering a coupon above LIBOR. A CPDO has first been issued by ABN Amro in 2006 called ABN Amro’s Surf 100. The structure of a CPDO has an important characteristic. A CPDO aims to generate the floating/fixed coupons by taking a leveraged position on a portfolio of credit default swap (CDS) indices [1]. CPDOs are issued by means of a Special Purpose Vehicle (SPV). The proceeds obtained from the holders of the CPDO at issuance are invested in a bank account. This amount is called collateral. Then, the leveraged position is taken on a notional index portfolio. In industry the notional size of the leveraged position is as follows:

\[ \ell_t^U = \max \left\{ \left( \frac{F_t - V_t}{PVS_t} \right) \times m, 0 \right\}, \]  

where \( F_t \) is the present value of the coupons and the principle, \( V_t \) is the current value of the assets hold by the CPDO manager, \( PVS_t \) is the current rate of the portfolio of CDS indices and \( m \) is a constant predetermined multiple. Everyday the leveraged position is dynamically adjusted according to mark to market gains and losses [3, 4].

In CPDO the gap between the present value of future obligations, \( F_t \) and that of the current assets, \( V_t \) of the CPDO manager is called “shortfall”. The leveraged position is directly affected from the shortfall. If there is no shortfall then the current value of the collateral is said to be enough to cover all the coupons and
the principle. Then, the leveraged position is closed and the manager only keeps the bank account. This case refers to as a “cash-in” event for CPDOs. Similarly, a “cash-out” event for CPDOs is the case in which the shortfall exceeds a given threshold. This case may also be considered as early default \[3,4,5\].

In this study we emphasize a model for the fair price of a CPDO. As observed from the studies performed by risk management agencies and in the literature about pricing and rating of CPDOs the models include too many variables. The implementation of the formulae is very challenging in decision mechanisms. In the study we combine the pricing models with the simple definition of the CPDO assets of the manager stated in optimal leverage papers.

In Section 2 we model the CPDOs. Then, we obtain the closed form formulae for the fair price of the CPDOs under the Laplace domain. Finally, in Section 3 we give a numerical analysis of the pricing equation.

2 Modelling and Pricing of CPDOs

In this part of the paper, we model the CPDO by considering all the risks embedded in the CPDO from the holders’ side.

In calculations we use continuous trading and we take the wealth definition stated in [2]. At issuance the CPDO manager pays some initial costs therefore we assume that the initial wealth is \( V_0 = v \leq G \). Then, the wealth dynamics are given as follows

\[
dV_t = rV_t dt + \ell_t dS_t - F_T (r + \nu) dt, \quad V_0 = v. \tag{2}
\]

In this equation, the first part of (2) shows the deposit account of the CPDO, and \( r, \nu \) represent the constant risk free interest rate and additional coupon promised by the CPDO. \( dS_t \) denotes the risks embedded in the portfolio of credit default swaps. The third part of (2) is the reduction part of the wealth due to the promised continuous coupon payments of the CPDO.

Moreover, \( F_t \) follows the differential equation

\[
dF_t = rF_t dt - G(r + \nu) dt, \quad F_0 = f. \tag{3}
\]

In Section 1 we see that the CPDO has three important features, ‘cash in’, ‘early default’ and ‘default on principal’. In the model we include all these features by considering the CPDO as a combination of a double barrier option with rebates and coupon payments. For this purpose, we define two stopping times, one for the \( early \ default \), \( \tau_{def} \) and one for closing the leveraged position, \( \tau_{str} \) for the CPDO, as

\[
\tau_{def} = \inf \{ t \in (0, T) \mid V_t < \beta_t \}, \tag{4}
\]

\[
\tau_{str} = \inf \{ t \in (0, T) \mid V_t \geq F_t \}, \tag{5}
\]

where \( \beta_t \) is either a predetermined constant or a variable default barrier. In other words:
i If \( t \geq \tau_{def} \), then the CPDO defaults in both coupon payments and final payment.

ii If \( t \geq \tau_{str} \), then the issuer has enough money to pay all the obligations. Therefore, the leveraged position is closed by putting \( \ell_s = 0 \) for all \( s \geq t \).

By assuming \( \tau_{def} > t \) the time \( t \) value of the coupons is modelled as follows:

\[
\psi(t) = \mathbb{E}^Q \left( e^{-r(T \wedge \tau_{def}) - t} \int_t^{(T \wedge \tau_{def})} G(r + \nu) e^{r(T \wedge \tau_{def}) - s} ds \bigg| \mathcal{F}_t \right)
\]

where \( Q \) presents the risk-neural probability measure.

The barrier option on the other side could be modelled as a barrier option on shortfall. If we define the shortfall as

\[
Y_t = F_t - V_t,
\]

and the default barrier, \( \beta_t = \alpha F_t \) where \( \alpha \in (0, 1) \) then the time \( t \) value of the barrier option is defined as follows:

\[
\phi(y, t) = \mathbb{E}^Q \left( \min \{G - y, G\} e^{-r(T - t)} \bigg| \mathcal{F}_t \right), \quad 0 \leq y \leq (1 - \alpha)F_t
\]

subject to the terminal and the boundary conditions

\[
\begin{align*}
\phi(y, T) &= \min \{G - y, G\}, \\
\phi(y, \tau_{def}) &= R, \\
\phi(y, \tau_{str}) &= Ge^{-r(T - \tau_{str})},
\end{align*}
\]

respectively. Since \( y \) is always positive – except on boundaries – we can write the terminal condition as \( \phi(y, T) = G - y \).

As a result the pricing equation for CPDOs is found as a summation of (8) and (6) which is

\[
P_{CPDO}(t) = \psi(t) + \phi(y, t).
\]

In the following we deal with the solution of the problem given in (8) under the assumptions of geometric Brownian and double exponential jump diffusion processes, respectively. In our calculations we refer to the work of Sepp [7] and we use the following properties of Laplace transformation

\[
\begin{align*}
L(x, p) &= \mathcal{L}(f(x, t)) = \int_0^\infty f(x, t)e^{-pt}dt, \\
\mathcal{L} \left( \frac{\partial f(x, t)}{\partial t} \right) &= pL(x, p) - f(x, 0), \\
\mathcal{L} \left( \frac{\partial^n f(x, t)}{\partial x^n} \right) &= \frac{\partial^n L(x, p)}{\partial x^n},
\end{align*}
\]

where \( \mathcal{L} \) denotes the Laplace transformation.

Moreover, we assume that the leverage factor is given as

\[
\ell_t = \max \left\{ \frac{(F_t - V_t)}{S_t} \times m, 0 \right\},
\]

3
where $m$ is a constant predetermined multiple.

In the Laplace domain we found the solution of $\phi(y,t)$ when the portfolio of credit default swaps follows a geometric Brownian motion with parameters $\mu$ and $\sigma$ as follows

$$L(x,p) = \begin{cases} (C_1 + C_3) e^{\xi_1 x} + C_3 e^{\xi_2 x} - \frac{G}{p} e^x + \frac{G}{r+p}, & \text{if } x < 0, \\ C_2 e^{\xi_2 x} + C_3 e^{\xi_1 x}, & \text{if } x \geq 0, \end{cases}$$

(13)

where $\xi_{1,2}$ are

$$\xi_{1,2} = -\frac{(r - \frac{1}{2} m^2 \sigma^2) \pm \sqrt{(r - \frac{1}{2} m^2 \sigma^2)^2 + 2 m^2 \sigma^2 (r + p)}}{m^2 \sigma^2}, \quad \xi_2 < 0 < \xi_1,$$

(14)

and

$$C_{1,2} = \frac{G}{\xi_1 - \xi_2} \frac{1 - \xi_{2,1}}{p}.$$  

(15)

Moreover, $C_3$ is given as

$$C_3 = e^{-\xi_1 x_d} \left( \frac{R}{p} - C_2 e^{\xi_2 x_d} \right),$$

(16)

and $L(x,p) = \mathcal{L}(\hat{\phi}(x,t)).$

Afterwards, we give the solution of $\phi(y,t)$ when the index portfolio follows a double exponential jump diffusion process. So, we assume a double exponential jump diffusion process for the index dynamics. Then, referring to [9,6] under the rational expectations economy and equivalent martingale measure $Q$, we obtain the dynamics for $Y_t$ as

$$dY_t = (r - \lambda^* \zeta^*) Y_t dt - m \sigma Y_t dW^*_t + \left( e^{J^*} - 1 \right) Y_t dN^*_t, \quad Y_0 = y,$$

(17)

where $W^*_t$ is a standard Brownian motion, $N^*_t$ is a Poisson process with intensity $\lambda^*$ and $W^*_t$, $N^*_t$, $J^*$ are independent under $Q$. Moreover, $J^*$ follows a modified, but a double exponential distribution

$$f_{J^*}(j) = q_1^* \frac{1}{\eta_{1}^*} e^{-\frac{j}{\eta_1^*}} \mathbb{I}_{j \geq 0} + q_2^* \frac{1}{\eta_{2}^*} e^{\frac{j}{\eta_2^*}} \mathbb{I}_{j < 0}, \quad 1 > \eta_{1}^* > 0, \quad \eta_{2}^* > 0,$$

(18)

where $q_1^*, q_2^* \geq 0$, $q_1^* + q_2^* = 1$, and $\zeta^* := \mathbb{E}^* \left[ e^{J^*} \right] + 1 = \frac{q_1^*}{1 - \eta_{1}^*} + \frac{q_2^*}{1 - \eta_{2}^*} + 1.$ In the solution procedure we apply the same methodology as in the geometric Brownian case. Then, the solution of the pricing equation in the Laplace domain that satisfies the boundary condition $L(x_d, p) = \frac{R}{p}$ is given by

$$L(x,p) = \begin{cases} (C_1 + C_5) e^{\xi_1 x} + (C_2 + C_6) e^{\xi_2 x} - \frac{G}{p} e^x + \frac{G}{r+p}, & \text{if } x < 0, \\ C_3 e^{\xi_2 x} + C_4 e^{\xi_1 x} + C_5 e^{\xi_1 x} + C_6 e^{\xi_2 x}, & \text{if } x \geq 0, \end{cases}$$

(19)
where $\xi_1, \xi_2, \xi_3,$ and $\xi_4$ are four distinct roots of the characteristic equation
\[
\frac{1}{2} m^2 \sigma^2 \xi^2 + \left( r - \lambda^* \xi^* - \frac{1}{2} m^2 \sigma^2 \right) \xi - \left( r + p + \lambda^* \right) + \lambda^* \left[ \frac{q_1^*}{1 - \eta_1 \xi} + \frac{q_2^*}{1 + \eta_2 \xi} \right] = 0, \tag{20}
\]
and they are ordered as
\[-\infty < \xi_4 < -\frac{1}{\eta_2} < \xi_3 < 0 < \xi_2 < \frac{1}{\eta_1} < \xi_1 < \infty.\]
Moreover, $C_1, C_2, C_3,$ and $C_4$ are obtained from the solution of the following system of equations
\[
\begin{bmatrix}
1 & 1 & -1 & -1 \\
\frac{1}{1 - \eta_1 \xi_1} & \frac{1}{1 - \eta_2 \xi_2} & -\frac{1}{\eta_1 \xi_3} & -\frac{1}{\eta_1 \xi_4} \\
\frac{1}{1 + \eta_2 \xi_1} & \frac{1}{1 + \eta_2 \xi_2} & -\frac{1}{\eta_1 \xi_3} & -\frac{1}{\eta_1 \xi_4}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix}
= \begin{bmatrix}
\frac{G}{p} - \frac{G}{r + p} \\
\frac{G}{p(1 - \eta_1)} - \frac{G}{r + p} \\
\frac{G}{p(1 + \eta_2)} - \frac{G}{r + p}
\end{bmatrix}, \tag{21}
\]
and $C_5, C_6$ are calculated from the system
\[
\begin{bmatrix}
e^{\xi_1 x_d} \\
\frac{1}{1 - \eta_1 \xi_1} e^{\left( \xi_1 - \frac{1}{\eta_1} \right) x_d} \\
\frac{1}{1 - \eta_2 \xi_2} e^{\left( \xi_2 - \frac{1}{\eta_1} \right) x_d} \\
0
\end{bmatrix}
\begin{bmatrix}
C_5 \\
C_6
\end{bmatrix}
= \begin{bmatrix}
\frac{p}{G} \eta_1 e^{\xi_1 x_d} - \sum_{i=3}^4 C_i \frac{q_i^*}{1 - \eta_i \xi_i} e^{\left( \xi_i - \frac{1}{\eta_i} \right) x_d}
\end{bmatrix}. \tag{22}
\]

3 An Application

We illustrate the behaviour of the risk neutral price of a constant proportion debt obligation via some numerical examples. In the application we use the Stehfest algorithm for inverse Laplace transformation stated in [8]. We particularly examine the dependence of the pricing equation on the volatility of the risky index and on the leverage multiplier $m$.

References


