An affine framework for the joint modelling of equity and credit risk

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Extended Abstract

1 Introduction

The last few years have witnessed an increasing popularity of hybrid equity/credit risk models. One of the most appealing features of such models is represented by their capability to link the stochastic behavior of the stock price (and of its volatility) with the random occurrence of the default event and, as a consequence, with the level of credit spreads. The relation between equity and credit risk is supported by strong empirical evidence (see the introductory sections of [1] and [3] for an overview of the related literature) and several studies document significant relationships between stock price volatility and credit spreads of corporate bonds and spreads of Credit Default Swaps.

In this paper, we propose a general framework for the joint modelling of equity and credit risk which allows for a flexible correlation structure between stock price, stochastic volatility and default intensity. The proposed framework is fully analytically tractable, since it relies on the powerful technology of affine processes, and nests several stochastic volatility models which have been proposed in the literature, thereby extending their scope to a defaultable setting. Furthermore, unlike the models proposed in [1], [2] and [3], we jointly consider both the historical and the risk-neutral probability measures and, by relying on the results of [5], we explicitly solve risk-management as well as pricing problems.

2 Modelling framework

Let \((\Omega, \mathcal{G}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\) be a given filtered probability space, where \(P\) denotes the physical (or historical) probability measure and \((\mathcal{F}_t)_{0 \leq t \leq T}\) is the filtration generated by an \(\mathbb{R}^d\)-valued Brownian motion \((W_t)_{0 \leq t \leq T}\). Let \(\tau : \Omega \rightarrow [0, T] \cup \{\infty\}\) be a random time which represents the default time of a given firm. We assume
that \( \tau \) is a doubly stochastic random time (in the sense of [4], Sect. 12.3.1) with stochastic \( P\)-intensity \( (\lambda_P^\tau)_{0 \leq t \leq T} \). Let the filtration \( (\mathcal{G}_t)_{0 \leq t \leq T} \) be the progressive enlargement of \( (\mathcal{F}_t)_{0 \leq t \leq T} \) with respect to \( \tau \) and let \( \mathcal{G} = \mathcal{G}_T \). Intuitively, the filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \) contains only the default-free market information, while the enlarged filtration \( (\mathcal{G}_t)_{0 \leq t \leq T} \) contains the full market information.

Let us denote by \( S_t \) the price at time \( t \in [0, T] \) of one share issued by the defaultable firm and let \( \tilde{S}_t \) be the corresponding pre-default value, i.e. \( S_t = 1_{\{\tau > t\}} \tilde{S}_t \), for all \( t \in [0, T] \). This corresponds to assuming that the stock price process jumps to zero as soon as the default event occurs and remains thereafter frozen at zero. Let \( (\tilde{v}_t)_{0 \leq t \leq T} \) denote the stochastic volatility of the stock and let \( (X_t)_{0 \leq t \leq T} \) be an \( \mathbb{R}^{d-2} \)-valued stochastic factor process which describes the evolution of the economy. Let also \( L_t := \log \tilde{S}_t \) and \( V_t := (\tilde{v}_t, X_t, L_t) \). We model the \( \mathbb{R}^d \)-valued process \( (V_t)_{0 \leq t \leq T} \) as the solution to the following SDE:

\[
dV_t = (AV_t + b) dt + \Sigma \sqrt{R_t} dW_t \quad V_0 = (v_0, X_0, \log S_0)
\]

with parameters \((A, b, \Sigma) \in \mathbb{R}^{d \times d} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}\) and where \( R_t \) is a diagonal \((d \times d)\)-matrix with elements given by \( R_t^{i,i} = \alpha_i + \beta_i^2 V_t \). Under suitable assumptions (see e.g. [4], Chpt. 10), there exists a unique strong solution to (1) on \( \mathbb{R}^{m}_+ \times \mathbb{R}^{d-m} \), for some \( m \in \{1, \ldots, d-1\} \). Furthermore, it can be easily shown that:

\[
dS_t = S_t \left( \bar{s} + \mu_1 \log S_t - \mu_2 v_t + \sum_{i=1}^{d-2} \gamma_i X^i_t \right) dt + S_t \sum_{i=1}^{d} \sqrt{\Sigma_{d,i} R_t^{i,i}} dW^i_t - S_t d1_{\{\tau \leq t\}}
\]

for suitable parameters \( \bar{s}, \mu_1, \mu_2 \) and \( \gamma \), thus giving rich and flexible dynamics to the defaultable stock price process. For simplicity, we suppose that the risk-free interest rate is deterministic and equal to \( r > 0 \). The \( P\)-intensity \( (\lambda_P^\tau)_{0 \leq t \leq T} \) of the default time \( \tau \) is modeled as follows:

\[
\lambda_P^\tau = \bar{\lambda}^P + (\Lambda_P)^\top V_t \quad \text{for all } t \in [0, T]
\]

with \( \bar{\lambda}^P \in \mathbb{R}_+ \) and \( \Lambda_P \in \mathbb{R}_+^m \times \{0\}^{d-m} \). Observe that the specification (1)-(3) allows for both direct and indirect interactions between the stock price process, its stochastic volatility and the default intensity. Note also that so far everything has been specified with respect to the original (historical) probability measure \( P \).

### 3 Risk-management applications

By relying on the affine structure of the general framework outlined in Section 2 we derive some simple results which can be of interest for risk-management purposes. As a preliminary, let us recall that for all \( t \leq T' \leq T \) and \( u \in \mathbb{C}^d \) (up to some technical conditions, see e.g. [4], Thm. 10.4):

\[
E^P \left[ e^{-\int_t^{T'} \lambda_P^\tau ds} e^{u^\top V_{T'}} \bigg| \mathcal{F}_t \right] = e^{\Phi^P(T' - t, u) + \Psi^P(T' - t, u)^\top V_t}
\]

(4)
where the functions $\Phi^P: [0, T] \times \mathbb{C}^d \to \mathbb{R}$ and $\Psi^P: [0, T] \times \mathbb{C}^d \to \mathbb{R}^d$ are given as solutions of a system of Riccati ODEs. As a first application, we can explicitly compute the $G_t$-conditional probability of surviving until $T' \in [t, T]$ by noting that

$$
P(\tau > T'|G_t) = \mathbb{1}_{\{\tau > t\}} E^P \left[ \exp \left( -\int_0^{T'} \lambda^P_s ds \right) \big| \mathcal{F}_t \right]
$$

and letting $u = 0$ in (4).

We show that many quantities of interest in view of risk-management applications can be computed explicitly after changing the measure from $P$ to $P_{T'}$, where $P_{T'}$ denotes the $T'$-survival measure, defined as follows:

$$
d_{P_{T'}}/dP := e^{-\int_0^{T'} \lambda^P_t dt} \left( E^P \left[ e^{-\int_0^{T'} \lambda^P_t dt} \right] - 1 \right) \mathbb{1}_{\{\tau \leq t\}} \gamma \tau
$$

for $T' \leq T$ (5)

Note that, for $t \leq T' \leq T$, the $F_t$-conditional characteristic function of $V_{T'}$ under the measure $P_{T'}$ can be easily obtained from (4). In particular, by relying on Fourier inversion techniques, we derive an explicit expression for the quantiles of the $G_t$-conditional distribution (under the historical probability $P$) of the defaultable stock price at a given future date $T' \leq T$ in terms of the $F_t$-conditional characteristic function of $V_{T'}$ under the $T'$-survival measure $P_{T'}$. This result will be important for the computation of Value-at-Risk and other risk measures.

4 Valuation of default-sensitive derivatives

Aiming at the valuation of default-sensitive financial derivatives, we need to shift our model from $P$ to some Equivalent Local Martingale Measure (ELMM) $Q$. It can be shown that all densities $dQ/dP$ admit the following representation:

$$
d_{P_{T'}}/dP := e^{-\int_0^{T'} \lambda^P_t dt} \left( E^P \left[ e^{-\int_0^{T'} \lambda^P_t dt} \right] - 1 \right) \mathbb{1}_{\{\tau \leq t\}} \gamma \tau
$$

where, due to (2), the risk-premia processes $(\theta_t)_{0 \leq t \leq T}$ and $(\gamma_t)_{0 \leq t \leq T}$ satisfy $P$-a.s. the following condition, for all $t \in [0, T \land \tau]$:

$$
\bar{s} + \mu_1 \log S_t - \mu_2 v_t + \sum_{i=1}^{d-2} \gamma_i X^i_t + \sum_{i=1}^d \Sigma_{d,i} \sqrt{R_{i,i}^t \theta^i_t} - \lambda^P_t (1 + \gamma_t) = r
$$

(7)

Note that the process $(\gamma_t)_{0 \leq t \leq T}$ represents the risk-premium associated to the default event and accounts for the non-diffusibility of default risk, see e.g. [6].

By relying on [5], we provide a full characterisation of all ELMMs which preserve the affine structure (in the sense that the specification (1)-(3) holds after the change of measure) by giving necessary and sufficient conditions on the processes $(\theta_t)_{0 \leq t \leq T}$ and $(\gamma_t)_{0 \leq t \leq T}$ satisfying (5)-(7). Working under an affine preserving ELMM will ensure analytical tractability under both the physical and the risk-neutral probability measures. Then, similarly as in Section [3] we show that many
pricing problems can be simplified by shifting the model to the $T'$-survival risk-neutral measure $Q_{T'}$, with:

$$
\frac{dQ_{T'}}{dQ} := e^{-\int_0^{T'} \lambda_s (1+\gamma_s) ds} \left( E^Q \left[ e^{-\int_0^{T'} \lambda_s (1+\gamma_s) ds} \right] \right)^{-1} \quad \text{for } T' \leq T \quad (8)
$$

Let us consider an European defaultable derivative with maturity $T' \leq T$ and payoff $F(\mathcal{V}_{T'})$ in the case of survival. Then we have the following pricing formula:

$$
E^Q \left[ e^{-r(T'-t)} F(\mathcal{V}_{T'}) \mathbf{1}_{\{\tau > T'\}} \bigg| \mathcal{G}_t \right] = e^{-r(T'-t)} 1_{\{\tau > t\}} e^{-\int_0^{T'-t} \lambda_s (1+\gamma_s) ds} \times E^{Q_{T'}} \left[ F(\mathcal{V}_{T'}) \bigg| \mathcal{F}_t \right] \quad (9)
$$

where the functions $\Phi^Q_t : \mathbb{R} 	imes \mathbb{C}^d \to \mathbb{R}$ and $\Psi^Q_t : \mathbb{R} \times \mathbb{C}^d \to \mathbb{R}^d$ are given as solutions of a system of Riccati ODEs. Since the $\mathcal{F}_t$-conditional characteristic function of $\mathcal{V}_{T'}$ under the measure $Q_{T'}$ can be obtained explicitly, the quantity $E^{Q_{T'}} \left[ F(\mathcal{V}_{T'}) \bigg| \mathcal{F}_t \right]$ can be computed in semi-closed form by relying on Fourier inversion techniques. In particular, we derive explicit expressions for the prices of corporate defaultable bonds and Call and Put options written on a defaultable stock together with a defaultable version of the classical Put-Call parity relation.

## 5 An example: the Heston with jump-to-default model

We illustrate the essential features of our general framework in the context of a simple example, which corresponds to a jump-to-default extension of the classical Heston stochastic volatility model. More specifically, with $d = 3$, we consider the following specification of (1):

$$
A = \begin{pmatrix} -k & 0 & 0 \\ 0 & -k_0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} k \bar{v} \\ k_0 x \\ \mu \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \bar{\sigma} & 0 & 0 \\ 0 & \sigma_0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \end{pmatrix}, \quad R_t = \begin{pmatrix} v_t & 0 & 0 \\ 0 & X_t & 0 \\ 0 & 0 & v_t \end{pmatrix}
$$

with $k \bar{v} \geq \sigma_0^2/2$, $k_0 x \geq \sigma_0^2/2$ and $\rho \in [-1, 1]$. The default $P$-intensity $(\lambda^P_t)_{0 \leq t \leq T}$ is specified as in (3). This specification extends the Heston jump-to-default model considered in (2) by allowing the default intensity to be a function of $v_t$ and of an additional stochastic factor $X_t$.

By specialising the general results of Sections 3-4 we numerically investigate the following issues:

(i) the impact of stochastic volatility and default risk on the $\mathcal{G}_t$-conditional distribution of the defaultable stock price;

(ii) the impact of default risk and of different specifications of the default intensity on the implied volatility surface of European vanilla options written on the defaultable stock.
Finally, by means of a simple example, we show that our model is particularly well-suited to the analysis of problems which involve simultaneously both the historical and the risk-neutral probability measures. We also discuss possible applications to the valuation of mortality-linked insurance products.

References


