Numerical Solution of Second-Order Ordinary Differential Equations by Improved Runge-Kutta Nyström Method of Order Five

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1 Introduction.

Consider the special second-order ordinary differential equations of the form

\[ y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0. \]  \hspace{1cm} (1)

Such problem often arise in science and engineering fields such as celestial mechanics, molecular dynamics, semi-discretization of wave equations and electronics. The second-order equations can be directly solved by using Runge-Kutta Nyström (RKN) methods or multistep methods [1, 2, 3, 4]. In this paper we developed the Improved Runge-Kutta Nyström (IRKN) method for solving equation (1). The methods are two step in nature and require lower number of function evaluations per step compared with the existing Runge-Kutta Nyström (RKN) methods. Therefore, the methods are computationally more efficient at achieving the higher order of local accuracy. Algebraic order conditions of the method are obtained and the fifth order method are derived with four stages. The numerical results are given to illustrate the efficiency of the proposed method compared to the existing RKN methods.

2 Construction of IRKN Method

We developed the IRKN method for solving the second-order equation directly by following the approach discussed in Dormand [1] on the derivation of RKN method.
and based on IRK method in [5, 6, 7]. The general form of explicit IRKN method with \( s \)-stages as follows:

\[
y_{n+1} = y_n + \frac{3h}{2} y'_n - \frac{h}{2} y'_{n-1} + h^2 \sum_{i=2}^{s} \bar{b}_i (k_i - k_{-i}),
\]

\[
y'_{n+1} = y'_n + h(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i})),
\]

\[
k_1 = f(x_n, y_n), \quad k_{-1} = f(x_{n-1}, y_{n-1}),
\]

\[
k_i = f(x_n + c_i h, y_n + h c_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j), \quad i = 2, \ldots, s,
\]

\[
k_{-i} = f(x_{n-1} + c_i h, y_{n-1} + h c_i y'_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} k_{-j})), \quad i = 2, \ldots, s.
\]  

(2)

2.1 Order Conditions

To find the order conditions for IRKN method we applied the Taylor’s series expansion to equations (2) (see [5, 6, 7]). Here, after using the Taylor’s series expansion the order conditions of method for \( y'_n \) and \( y_n \) up to order five are presented in Table 1.

<table>
<thead>
<tr>
<th>order of method</th>
<th>order condition for ( y' )</th>
<th>order condition for ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>first order</td>
<td>( b_1 - b_{-1} = 1 )</td>
<td></td>
</tr>
<tr>
<td>second order</td>
<td>( b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} )</td>
<td></td>
</tr>
<tr>
<td>third order</td>
<td>( \sum_{i=2}^{s} b_i c_i = \frac{5}{12} )</td>
<td>( \sum_{i=2}^{s} \bar{b}_i = \frac{5}{12} )</td>
</tr>
<tr>
<td>fourth order</td>
<td>( \sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3} )</td>
<td>( \sum_{i=2}^{s} \bar{b}_i c_i = \frac{1}{6} )</td>
</tr>
<tr>
<td>fifth order</td>
<td>( \sum_{i=2}^{s} b_i c_i^3 = \frac{31}{120} )</td>
<td>( \sum_{i=2}^{s} \bar{b}_i c_i^2 = \frac{31}{360} )</td>
</tr>
<tr>
<td></td>
<td>( \sum_{i=2}^{s} b_i a_{ij} c_j = \frac{31}{720} )</td>
<td></td>
</tr>
</tbody>
</table>
2.2 Fifth order IRKN method (IRKN5)

The IRKN5 method with four-stages \( s = 4 \) is given as:

\[
y_{n+1} = y_n + \frac{3h}{2} y'_n - \frac{h}{2} y'_{n-1} + h^2 \sum_{i=2}^{4} \bar{b}_i (k_i - k_{-i}),
\]

\[
y'_n = y'_n + h(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{4} b_i (k_i - k_{-i})),
\]

\[
k_1 = f(x_n, y_n),
\]

\[
k_{-1} = f(x_{n-1}, y_{n-1}),
\]

\[
k_i = f(x_n + c_i h, y_n + h c_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j), \quad i = 2, 3, 4,
\]

\[
k_{-i} = f(x_{n-1} + c_i h, y_{n-1} + h c_i y'_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} k_{-j}), \quad i = 2, 3, 4.
\]

(3)

To find the coefficients of IRKN5 method in equations (3), order conditions in Table 1 up to order five for \( y_n \) and \( y'_n \) and row sum conditions must be satisfied. Therefore we need to satisfy the following equations:

\[
b_1 - b_{-1} = 1, \quad (4)
\]

\[
b_{-1} + b_2 + b_3 + b_4 = \frac{1}{2}, \quad (5)
\]

\[
b_2 c_2 + b_3 c_3 + b_4 c_4 = \frac{5}{12} \quad (6)
\]

\[
b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{3}, \quad (7)
\]

\[
b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = \frac{31}{120}, \quad (8)
\]

\[
b_3 a_{32} c_2 + b_4 (a_{42} c_2 + a_{43} c_3) = \frac{31}{720}, \quad (9)
\]

\[
\bar{b}_2 + \bar{b}_3 + \bar{b}_4 = \frac{5}{12}, \quad (10)
\]

\[
\bar{b}_2 c_2 + \bar{b}_3 c_3 + \bar{b}_4 c_4 = \frac{1}{6}, \quad (11)
\]
\[ \tilde{b}_2c_2^2 + \tilde{b}_3c_3^2 + \tilde{b}_4c_4^2 = \frac{31}{360}, \]  
\( (12) \)

\[ \frac{1}{2}c_2^2 = a_{21}, \]  
\( (13) \)

\[ \frac{1}{2}c_3^2 = a_{31} + a_{32}, \]  
\( (14) \)

\[ \frac{1}{2}c_4^2 = a_{41} + a_{42} + a_{43}. \]  
\( (15) \)

We choose \( c_2 = \frac{1}{4}, \ c_3 = \frac{1}{4} \) and \( c_4 = \frac{3}{4} \) as free parameters and obtained the remaining parameters. By solving equations (4) - (8) and (10) - (12) we have:

\[ b_{-1} = \frac{1}{45}, \ b_1 = \frac{46}{45}, \ b_2 = -\frac{1}{15}, \ b_3 = -\frac{1}{10}, \ b_4 = \frac{29}{45}. \]  
\( (16) \)

\[ \tilde{b}_2 = \frac{49}{180}, \ \tilde{b}_3 = \frac{7}{180}, \ \tilde{b}_4 = \frac{19}{180}. \]  
\( (17) \)

By solving equations (13)-(14) we have:

\[ a_{21} = \frac{1}{32}, \ a_{32} = \frac{1}{8} - a_{31}. \]  
\( (18) \)

We set \( a_{41} = \frac{4}{32}, \ a_{42} = 0 \) and after substituting into (15) we obtain \( a_{43} = \frac{5}{32} \). Finally by substituting obtained parameters into (9) we find the value of \( a_{32} = \frac{7}{24} \), therefore all coefficients of IRKN5 are shown in Table 2.

| Table 2: Table of coefficients of IRKN5 |
|---|---|---|---|---|
| 0 | \( \frac{1}{4} \) | \( \frac{1}{32} \) |
| \( \frac{1}{4} \) | \( \frac{1}{4} \) | \( \frac{46}{45} \) | \( -\frac{1}{15} \) | \( -\frac{1}{10} \) | \( \frac{29}{45} \) |
| \( \frac{3}{4} \) | \( 0 \) | \( \frac{5}{32} \) |
| \( \frac{1}{15} \) | \( \frac{1}{180} \) | \( \frac{7}{180} \) | \( \frac{19}{180} \) | \( \frac{49}{180} \) |
3 Numerical Examples

In this section, we tested a standard set of second-order initial value problems to show the efficiency and accuracy of the proposed method. The exact solution \( y(x) \) and \( y'(x) \) are used for starting values of \( y_1 \) and \( y'_1 \) at the first step \([x_0 \ x_1]\). The following problems are solved for \( x \in [0 \ 10] \).

Problem 1 (The undamped Duffing’s equations [2])

\[
y'' + y + y^3 = 0.002 \cos(1.01x), \quad y(0) = 0.200426728067, \quad y'(0) = 0.
\]
exact solution computed by Galerkin method and given by:

\[
y(x) = \sum_{i=0}^{4} a_{2i+1} \cos[1.01(2i + 1)x],
\]

with \( a_1 = 0.200179477536, \ a_2 = 0.24 \times 10^{-3}, \ a_5 = 0.304014 \times 10^{-6}, \ a_7 = 0.374 \times 10^{-9} \) and \( a_9 < 10^{-12} \).

Problem 2 (An almost periodic Orbit problem studied by Stiefel and Bettis [3])

\[
y'' + y = 0.001 e^{ix}, \quad y(0) = 1, \quad y'(0) = 0.9995i.
\]

exact solution: \( y(x) = (1 - 0.0005ix)e^{ix} \). we write in equivalent form

\[
y'' + y_1 = 0.001 \cos(x), \quad y_1(0) = 1, \quad y'_1(0) = 0,
\]
\[
y'' + y_2 = 0.001 \sin(x), \quad y_2(0) = 0, \quad y'_2(0) = 0.9995.
\]

exact solution: \( y_1(x) = \cos(x) + 0.0005x \sin(x), \ y_2(x) = \sin(x) - 0.0005x \cos(x) \).

To illustrate the efficiency of new methods we compared the numerical results with existing methods. The codes have been denoted by the following.

(i) IRKN5 : The Improved Runge-Kutta Nystrom method of order five with four stages in this paper.

(ii) RKNV3 : The third order Runge-Kutta Nystrom method with zero dissipation three stages given in van der Houwen and Sommeijer [4]

(iii) RKND3 : The third order three stages Runge-Kutta Nystrom method given in Dormand.[1]

(iv) RKNV4 : The fourth order Runge-Kutta Nystrom method with ten order dispersion, fifth order dissipation four stages given in Van der Huwen and Sommeijer [4]

(v) RKN4 : The fourth order three stages classical Runge-Kutta Nystrom method given in Garcia et al .[8]

In Tables 3 and 4 the maximum global error against the different values of step size \( h \) is presented for tested problems and we observed that the new method is more accurate compared to the given methods.
Table 3: Maximum global error versus step size $h$ for IRKN5 and RKN methods with $s$-stages for test problem 1

<table>
<thead>
<tr>
<th>Method (s-stages)</th>
<th>$h=0.1$</th>
<th>$h=0.01$</th>
<th>$h=0.005$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRKN5 (s=4)</td>
<td>6.07 E-8</td>
<td>1.01 E-11</td>
<td>1.07 E-11</td>
</tr>
<tr>
<td>RKNV3 (s=3)</td>
<td>2.37 E-3</td>
<td>2.38 E-5</td>
<td>5.09 E-6</td>
</tr>
<tr>
<td>RKND3 (s=3)</td>
<td>3.01 E-5</td>
<td>3.01 E-8</td>
<td>3.77 E-9</td>
</tr>
<tr>
<td>RKNV4 (s=4)</td>
<td>3.36 E-6</td>
<td>8.11 E-10</td>
<td>5.13 E-10</td>
</tr>
<tr>
<td>RKNC4 (s=3)</td>
<td>6.68 E-7</td>
<td>8.41 E-11</td>
<td>3.48 E-11</td>
</tr>
</tbody>
</table>

Table 4: Maximum global error versus step size $h$ for IRKN5 and RKN methods with $s$-stages for test problem 2

<table>
<thead>
<tr>
<th>Method (s-stages)</th>
<th>$h=0.1$</th>
<th>$h=0.01$</th>
<th>$h=0.005$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRKN5 (s=4)</td>
<td>2.86 E-7</td>
<td>2.89 E-12</td>
<td>8.98 E-14</td>
</tr>
<tr>
<td>RKNV3 (s=3)</td>
<td>1.07 E-2</td>
<td>1.07 E-4</td>
<td>2.69 E-5</td>
</tr>
<tr>
<td>RKND3 (s=3)</td>
<td>1.49 E-4</td>
<td>1.18 E-7</td>
<td>1.85 E-8</td>
</tr>
<tr>
<td>RKNV4 (s=4)</td>
<td>3.09 E-6</td>
<td>3.05 E-10</td>
<td>1.09 E-11</td>
</tr>
<tr>
<td>RKNC4 (s=3)</td>
<td>1.40 E-5</td>
<td>3.74 E-9</td>
<td>2.47 E-9</td>
</tr>
</tbody>
</table>
4 Conclusion

In this paper we constructed the Runge-Kutta Nystrom (RKN) method for solving second order ODEs. The order conditions of the IRKN methods are derived and using these order conditions, the methods are obtained for order 5 with 4 stages. Numerical results are presented to illustrate the efficiency of IRKN5 compared to the existing methods. The IRKN5 is more efficient compared to RKNV3, RKND3, RKNV4 and RKN4 methods.

References


