Optimal control problems (OCP) for Burgers equation are important for the development of numerical methods for optimal control of more complicated models in fluid dynamics such as Navier-Stokes equations.

In this work, we analyze variant preconditioners for the saddle point problem arising from boundary control of unsteady Burgers equation. As for a solution approach for the discretization of the OCPs, we use discretize-then-optimize. In the discretize-then-optimize approach the state equation is discretized and then the optimality system for the finite dimensional optimization problem is derived. This approach is also referred to as the black-box approach. In other words, an existing algorithm for the solution of the state equation is embedded into an optimization loop. The black-box approach is easy to use because it requires no modification to an existing partial differential equation (PDE) integrator. As in the case of Burgers equation, the repeated costly solution of the state equation is needed.

Treating the control and state as independent of optimization variables the discrete optimality conditions yield a system

\[
\begin{pmatrix}
E & L^T \\
L & 0
\end{pmatrix}
\begin{pmatrix}
x
\end{pmatrix}
=
\begin{pmatrix}
b
\end{pmatrix}
\]

This system is usually a kind of saddle point problem, where \( A \) is symmetric and has sparse structure. This kind of system usually requires preconditioning. There have been many works concerning block iterative solutions of \( Ax = b \) \[2, 3, 4, 5\]. Recently, Wathen, Stoll and Rees have made many researches related to all-at-once preconditioning of linear control problems \[6, 7, 8, 9, 10\]. The structure of the control constrained problems are covered. They showed how to handle control constraints. Different preconditioning methods were covered. Especially elliptic problems were worked since it is easy to implement without having memory problems. However, considering parabolic problems if every time step is considered in a block matrix then the system \( Ax = b \) is obtained. This is done for heat equation in \[4, 9\]. For the non-linear control problems it is not as easy as in the linear case.
We consider the following optimal control problem for the unsteady Burgers equation with Robin type boundary controls

\[
\min J(y, u, v) = \frac{\alpha}{2} \int_Q (y - y_d)^2 dx dt + \frac{1}{2} \int_0^T \beta_u |u|^2 + \beta_v |v|^2 dt
\]

subject to

\[
\begin{align*}
y_t + y y_x - \nu y_{xx} &= f \quad \text{in } Q, \\
\nu y_x(\cdot, 0) + \sigma_0 y(\cdot, 0) &= u \quad \text{in } (0, T), \\
\nu y_x(\cdot, 1) + \sigma_1 y(\cdot, 1) &= v \quad \text{in } (0, T), \\
y(0, \cdot) &= y_0 \quad \text{in } \Omega,
\end{align*}
\]

where, \( T > 0 \) denotes the fixed time, \( Q = (0, T) \times \Omega \) with \( \Omega = (0, 1) \) and \( \nu > 0 \) is the viscosity parameter. The given desired state is \( y_d \in L^2(Q) \), whereas \( u, v \) are the control variables, \( \beta_u, \beta_v \) are positive constants, \( \sigma_0, \sigma_1 \in L^\infty(0, T) \), \( y_0 \in L^2(\Omega) \) and \( f \in L^2(Q) \) is the forcing function and \( \alpha \in L^\infty(Q) \) is the regularization parameter.

As for comparison, Newton-type linearization procedures are considered. Two commonly used iterative solution techniques for Burgers equation are described by

\[
y_{t}^{k+1} + \tau y^{k+1} y_x + y^k y_x^{k+1} - \nu y_{xx}^{k+1} = \tau y^k y_x + f
\]

where \( y^k \) denote the \( k \)-th iterate and \( \tau \in \{0, 1\} \) defines the linearization method: \( \tau = 1 \) for the Newton method, and \( \tau = 0 \) for Picard iteration. While the Newton method uses an exact linearization of the discrete equations, the Picard iteration can be regarded as using an inexact linearization. The linear systems of the Picard method are easier to solve, but this comes at the cost of slower convergence of the nonlinear system. However, since Picard iteration is globally convergent it can be used to provide a good initial iterate for the Newton method, which is not globally convergent. For \( \tau = 1 \), we write

\[
y_t - \nu y_{xx} + (\bar{y} y)_x = \bar{y} y_x + f
\]

We use the Galerkin finite element method for space discretization. The state \( y \), linearized state \( \bar{y} \) are discretized by using standard Galerkin method with linear finite elements on the interval \((0, 1)\) with \( n \) uniform subdivisions.

\[
y(x, t) \sim \sum_{j=0}^{n} y_j(t) \phi_j(x) \text{ and } \bar{y}(x, t) \sim \sum_{k=0}^{n} \bar{y}_k(t) \phi_k(x).
\]

We define

\[
y = (y_1(t), \ldots, y_{n+1}(t)), \quad v = (0, \ldots, 0, v(t)), \quad u = (u(t), 0, \ldots, 0)
\]

and \( \bar{y} = (\bar{y}_1(t), \ldots, \bar{y}_{n+1}(t)) \).
Then, the semi-discrete control problem follows
\[
\min \quad J_h = \int_0^T \frac{1}{2} (y - y_d)^T M (y - y_d) dt + \int_0^T \beta_u |u|^2 + \beta_v |v|^2 dt
\]
\[
\text{s.t.} \quad M y_t + S y + q(y) = f_h(u, v), \quad y(0) = y_0.
\]

We use semi-implicit time approximation which consists in evaluating the diffusive part \(y_{xx}\) at the time level \(i + 1\), whereas the remaining parts are considered at time level \(i\). When this scheme is applied to a non-linear advection, it provides an efficient linearization. We define
\[
Y = (y^1, \ldots, y^N) \text{ and } U = (u^1, \ldots, u^N), \text{ and } V = (v^1, \ldots, v^N)
\]

The optimality system containing first order optimality conditions is obtained by introducing the extended Lagrangian containing the Lagrange multiplier \(P\).
\[
\nabla_Y L(Y, U, V, P) = \Delta t M_{1/2} (Y - Y_d) - K^T P = 0,
\]
and,
\[
\nabla_P L(Y, U, V, P) = -KY + \Delta t L_2 V - \Delta t L_1 U + Q + d = 0,
\]
with the gradient equations
\[
\beta_u \Delta t L_1 U - \Delta t L_1 P = 0,
\]
and
\[
\beta_v \Delta t L_2 V + \Delta t L_2 P = 0.
\]

The optimality system can be written as
\[
\begin{pmatrix}
\Delta t M & 0 & 0 & -K^T \\
0 & \beta_u \Delta t L_2 & 0 & \frac{\Delta t}{2} L_2 \\
-\mathcal{K} & \frac{\Delta t}{2} L_2 & \beta_u \Delta t L_1 & -\frac{\Delta t}{2} L_1
\end{pmatrix}
\begin{pmatrix}
Y \\
V \\
U \\
P
\end{pmatrix}
= \begin{pmatrix}
M_{1/2} Y_d \\
0 \\
0 \\
Q + d
\end{pmatrix},
\]
where \(\mathcal{M}, \mathcal{K}, Q, d, L_1, \text{ and } L_2\) are related block matrices.

Combining all time-step solution in a block matrix leads to an indefinite saddle point problem, which is usually solved iteratively using preconditioners. We note that when \(A\) is nonsingular then the following block triangular factorization holds
\[
\begin{pmatrix}
E & L^T \\
L & 0
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
LE^{-1} & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
S & 0 \\
0 & I
\end{pmatrix},
\]
where \(S = LE^{-1}L^T\) is the Schur complement of \(E\). This factorization gives an idea about preconditioning. As for solution method various preconditioned Krylov subspace methods or multilevel methods can be considered to increase the convergence rate [1, 2, 3]. Among the Krylov subspace methods there are several preconditioning techniques:
• CG, MINRES, and SYMMLQ require positive semi definiteness,
• SQMR requires symmetric matrix,
• GMRES.

We propose the following preconditioner

\[ P = \begin{pmatrix}
\Delta t M & 0 & 0 & 0 \\
0 & \beta_c \Delta t L_2 & 0 & 0 \\
0 & 0 & \beta_c \Delta t L_1 & 0 \\
0 & 0 & 0 & S \\
\end{pmatrix} \]

Numerical results with varying preconditioners are presented.

References


