Solutions of Initial and Boundary Value Problems by the Variational Iteration Method

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\section{Introduction.}

The Variational Iteration Method (VIM) is an iterative method that approximates solutions of differential equations. The method is a modification of the so-called general Lagrange multipliers \cite{2}. The method is based on the idea of constructing a correction functional that uses a Lagrange multiplier.

In this study, we analyse some basic properties of the Lagrange multiplier; and using these, we propose a new algorithm for solving initial and boundary value problems. In \cite{1}, a new approach that uses the matrix valued Lagrange multiplier has been proposed. Further it is proven that there exists a close relation between the Lagrange multiplier and fundamental matrices of the homogeneous systems.

This relation leads us to prove that whenever the initial approximation satisfies the initial conditions, the solution of initial value problem can be obtained with a single iteration. Moreover, it is also shown that when initial approximation satisfies the boundary conditions of the linear boundary value problem, the solution of the system can be obtained with a single iteration, too. Main advantage of such a result is that the modified algorithm can be extended to nonlinear boundary value problems, and further, an approximate solution of such problems can be obtained without using the theory of Green’s functions.

\section{Solution of Initial Value Problem}

We consider the \textit{mth} order linear homogeneous differential equation

\[ L_{m,t}x(t) := p_0(t)x^{(m)}(t) + p_1(t)x^{(m-1)}(t) + \cdots + p_m(t)x(t) = 0 \quad (1) \]

subject to the initial conditions

\[ x(t_0) = \alpha_1, \; x'(t_0) = \alpha_2, \; \ldots, \; x^{(m-1)}(t_0) = \alpha_m, \]
where \( p_0(t) > 0 \) for all \( t \in I \), and \( \dot{x}^{(i)} \) represents the \( i \)th derivative \( d^i x(t)/dt^i \) with respect to \( t \).

The VIM constructs the correction functional in the following form

\[
x_{n+1}(t) = x_n(t) + \int_{t_0}^{t} \lambda(s; t)L_{m,s}x_n(s)ds,
\]

where \( x_n(t) \) is named as the \( n \)th order approximation and \( \lambda(s; t) \) is the so-called Lagrange multiplier. An easy calculation shows that

\[
L_{m,s}^\dagger \lambda(s; t) = 0
\]

subject to the conditions

\[
\frac{\partial^j \lambda(s; t)}{\partial s^j} \bigg|_{s=t} = 0, \quad j = 0, 1, 2, \ldots, m - 2, \quad \frac{\partial^{m-1} \lambda(s; t)}{\partial s^{m-1}} \bigg|_{s=t} = \frac{(-1)^m}{p_0(t)}.
\]

Here, \( L_{m,s}^\dagger \) represents the adjoint linear differential (operator) expression

\[
L_{m,s}^\dagger(\cdot) = (-1)^m \frac{d^m}{ds^m} (p_0(s) \cdot) + (-1)^{m-1} \frac{d^{m-1}}{ds^{m-1}} (p_1(s) \cdot) + \cdots + (p_m(s) \cdot).
\]

On the other hand, one can rewrite (1) as

\[
\dot{x} = A(t)x
\]

where

\[
A = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-p_m & -p_{m-1} & \cdots & -p_1
\end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{pmatrix}
\]

with \( \tilde{p}_i = p_i(t)/p_0(t) \). In this case, the correction functional is

\[
x_{n+1}(t) = x_n(t) + \int_{t_0}^{t} \Lambda(s; t) \left\{ (x_n(s))' - A(s)x_n(s) \right\} ds.
\]

Here, \( \Lambda(s; t) \) is an \( m \times m \) matrix-valued Lagrange multiplier, unlike the scalar multiplier \( \lambda(s; t) \). From [1], we have

\[
\Lambda(s; t) = -\Psi^T(s; t)
\]

where \( \Psi(s) \) represents the fundamental matrix of the adjoint equation for (5):

\[
y' = -A^T(s)y, \quad y = (y_1, y_2, \ldots, y_m)^T.
\]
Furthermore, it can easily be shown that

\[ \lambda(s; t) = \Lambda_1(m; s), \tag{10} \]

which provides us with a new method for finding the Lagrange multiplier.

By using the properties of \( \Lambda(s; t) \), it turns out that

\[ L_{m,t} \lambda(s; t) = 0, \tag{11} \]

with

\[ \frac{\partial^j \lambda(s; t)}{\partial t^j} \bigg|_{t=s} = 0, \quad j = 0, 1, \ldots, m-2, \]

\[ \frac{\partial^{m-1} \lambda(s; t)}{\partial t^{m-1}} \bigg|_{t=s} = -\frac{1}{p_0(s)}. \]

Note that the derivatives are with respect to \( t \), but not \( s \). Hence, we may state and prove the following theorem.

**Theorem 2.1.** Let

\[ L_{m,t} x(t) = f(t) \tag{12} \]

and the initial conditions be

\[ x(t_0) = \alpha_1, \dot{x}(t_0) = \alpha_2, \ldots, \dot{x}^{(m-1)}(t_0) = \alpha_m. \]

Then, for any sufficiently smooth \( x_0 \) that satisfies the initial conditions,

\[ x_1(t) = x_0(t) + \int_{t_0}^{t} \lambda(s; t) \left\{ L_{m,s} x_0(s) - f(s) \right\} ds \]

is the unique solution of (12).

The question is that ‘Can the same idea be applied to the boundary value problems?’

### 3 Solution of Boundary Value Problems

Consider

\[ S x(t) = f(t) \tag{13} \]

where the operator \( S \) is defined by the linear differential expression

\[ L_{m,t} x(t) := p_0(t) \dot{x}^{(m)}(t) + p_1(t) \dot{x}^{(m-1)}(t) + \cdots + p_m(t)x(t) \tag{14} \]

on the space of functions \( C^m(I, \mathbb{R}) \) satisfying

\[ U_v(x) = \sum_{j=1}^{m} M_{v,j} x^{(j-1)}(a) + N_{v,j} x^{(j-1)}(b) = 0, \quad v = 1, 2, \ldots, m. \tag{15} \]
Equivalently, we write (15) as

$$U(x) = M\hat{x}(a) + N\hat{x}(b),$$

(16)

where $$U = (U_1, U_2, \ldots, U_m)^T$$, $$\hat{x} = (x_1, x_2, \ldots, x_m)^T$$.

We assume that

$$L_{m,t}x = 0, \quad U(x) = 0,$$

(17)

has only the trivial solution: $$x \equiv 0$$. So, the unique solution of (13) can be written in the form

$$x(t) = \int_a^b G(t, s)f(s)ds,$$

where $$G(t, s)$$ denotes the Green’s function of the operator $$S$$.

Thus, application of VIM yields the following theorem.

**Theorem 3.1.** Let $$x_0 \in C(I, \mathbb{R})$$ satisfy the boundary conditions given in (15), then

$$x_1(t) = \tilde{x}_1(t) - y_1(t)$$

(18)

solves the boundary value problem (13), where

$$\tilde{x}_1(t) = x_0(t) + \int_a^t \lambda(s; t) \left\{ L_{m,s}x_0(s) - f(s) \right\} ds$$

and $$y_1$$ is any function that satisfies $$L_{m,t}y_1 = 0$$, and $$U(y_1) = U(\tilde{x}_1)$$.

4 Application and Conclusion

To put all these in practice and for practical work, we modify the VIM according to the theorems above: we applied the VIM to initial and boundary value problems arising from different applications. Theoretical results are also validated by numerical computations.

Furthermore, we propose a new algorithm, based on the theorems above, especially for nonlinear boundary value problems.

References
