Efficient Simulation of a Multi-factor Stochastic Volatility Model

Ahmet Görçü¹, Giray Ökten²

¹ Department of Mathematical Sciences, Xi’an Jiaotong-Liverpool University, Suzhou, 215123, China  
² Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA

1 Introduction.

Volatility is one of the most important factors in the pricing of financial derivatives. Although the standard Black-Scholes model with the constant volatility assumption serves as a good basis for modeling the asset price dynamics, there has been a wide-range of research done to improve upon this model. Black-Scholes model has been extended to incorporate either constant volatility plus jumps or stochastic volatility with/without jumps to provide a better fit of implied volatility surfaces. In particular, a mean-reverting stochastic process is suitable for modeling volatility. Mean-reversion refers to a linear pull back term in the mean growth rate of the volatility process. This class of models is known as stochastic volatility models. One approach within the class of stochastic volatility models is to consider volatility driven by two factors; the mean reversion of volatility is captured using two stochastic processes one with a slow mean reversion and the other with a fast mean reversion property. Under this framework Fouque et al. [5] use a combination of singular and regular perturbations to derive a closed form formula to approximate European option prices. In an earlier work by Fouque et al. [3] fast mean reverting volatility is considered. In another study by Christoffersen et al. [1] a two factor stochastic volatility model is used and the use of two factors is justified with principle component analysis of the Black-Scholes implied volatility surface. Principle components analysis shows that the first two eigenvalues account for more than 95% of the total variation in the volatility process. This shows that only a few number of factors would be sufficient to explain most of the variation in the volatility process. However, mean reverting parameters are not categorized as fast and slow mean reverting as it is done in [2], [4], and [5]. Market data shows the need for introducing also a slowly varying factor into the model for stochastic volatility. In particular, for options with longer maturities the addition of a slow varying factor is useful to improve the fit. The calibration to the market data is also
presented in Fouque et al. \cite{4} with supporting evidence in favor of using fast and slow mean reversion parameters.

Empirical evidence suggests that multi-factor stochastic volatility models provide significant improvements in explaining the term structure of volatility. Under the multi-factor stochastic volatility models Monte Carlo simulation methods can be used to evaluate option prices efficiently. Fouque and Han \cite{2} propose the use of variance reduction techniques such as importance sampling and control variate methods to evaluate risk neutral pricing of financial derivatives in a multi-factor volatility model.

In this paper we consider the same two-factor stochastic volatility model given in \cite{2} and compare the efficiency of different estimators for two option pricing problems, namely European and Barrier call options. Given the same model setup, we derive the asymptotic approximation formula for the Barrier option and derive the corresponding importance sampling estimator by the use of zeroth and first order terms in the asymptotic expansion of the pricing function. Our results show that using an importance sampling estimator with the first order expansion is not efficient compared to the importance sampling estimator derived from the zeroth order term.

Furthermore, numerical efficiency is significantly improved by using randomized quasi-Monte Carlo in addition to the importance sampling method. We improved the mean square error by a factor of 10 to 40 by using importance sampling together with linear scrambled Faure sequence, compared to the crude Monte Carlo estimator. For low dimensional problems it is well known that the quasi-Monte Carlo method can improve the accuracy of estimates. Given the high dimensionality (i.e. \(d = 600\)) of the problems we considered, the effectiveness of randomized quasi-Monte Carlo method should be noted.

\section{Two-Factor Stochastic Volatility Model}

In the study by Fouque et al. \cite{5}, a combination of singular and regular perturbations are used to derive the asymptotic approximation formula for the vanilla European call option price. A class of multi-factor volatility models has been introduced which are driven by two diffusions; one fluctuating on a fast time scale and another one fluctuating on a slower time scale. It has been shown that it is possible to combine a singular perturbation expansion with respect to the fast scale with a regular perturbation expansion with respect to the slow scale.

Following the multi-factor stochastic volatility model in \cite{2} and \cite{5}, we consider a family of stochastic volatility models, where \(S_t\) is the underlying price, \(Y_t\) evolves as an Ornstein-Uhlenbeck (OU) process, and \(Z_t\) follows another diffusion process. Given the risk neutral probability measure \(\mathbb{P}^*\), the stochastic volatility
model is described with the following equations:

\[ dS_t = rS_t dt + \sigma_t S_t dW_t^0 \]  
\[ \sigma_t = f(Y_t, Z_t) \]  
\[ dY_t = \left( \alpha(m_f - Y_t) - \nu_f \sqrt{2\alpha_\Lambda(Y_t, Z_t)} \right) dt + \nu_f \sqrt{2\alpha} \left( \rho_1 dW_t^0 + \sqrt{1 - \rho_1^2} dW_t^1 \right) \]  
\[ dZ_t = \left( \delta(m_s - Z_t) - \nu_s \sqrt{2\delta_\Lambda(Y_t, Z_t)} \right) dt + \nu_s \sqrt{2\delta} \left( \rho_2 dW_t^0 + \rho_{12} dW_t^1 + \sqrt{1 - \rho_2^2 - \rho_{12}^2} dW_t^2 \right) \].

Here \((W_t^0, W_t^1, W_t^2)\) are independent standard Brownian motions with instant correlation coefficients \(\rho_1, \rho_2,\) and \(\rho_{12}\) such that \(\rho_1^2 < 1,\) and \(\rho_2^2 + \rho_{12}^2 < 1,\) whereas \(S_t\) is the underlying stock price process with the risk free rate equal to \(r.\)

The volatility process \(\sigma_t\) is driven by two factors: \(Y_t\) and \(Z_t.\)

3 Asymptotic option price approximation and importance sampling

Under the two factor stochastic volatility model a pointwise price approximation is derived for a European call option in [5]. The formula for \(P^{\epsilon,\delta},\) i.e. the price of a European option with payoff function \(H,\) is given as

\[ P^{\epsilon,\delta}(t, x, y, z) \approx \tilde{P}(t, x, z), \]

where

\[ \tilde{P} = P_{BS(\bar{\sigma})} + (T - t) \times \left( V_0 \frac{\partial}{\partial \sigma} + V_1 x \frac{\partial^2}{\partial x \partial \sigma} + V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2}{\partial x^2} \right) \right) P_{BS(\bar{\sigma})},\]

with an accuracy of order \( (\epsilon | \log \epsilon | + \delta) \) for call options. The price \(P_{BS(\bar{\sigma})}\) is the homogenized price which solves the Black-Scholes equation and \(\bar{\sigma}\) is the effective volatility.

The asymptotic approximation formula in Equation (5) is a first order approximation to the option price under the two-factor stochastic volatility model. For the European call option pricing the zeroth order term in Equation (5) is simply given by the Black-Scholes option price. However, obtaining the first order approximation requires the computation of various partial derivatives of the Black-Scholes price and the parameters \((V_0, V_1, V_2, V_3),\) which are given in Equations (19)-(22) in [2].
We consider mainly two alternative importance sampling estimators: one derived by using the zeroth order approximation and the other one derived by using the first order asymptotic approximation. These estimators are compared with the benchmark of crude Monte Carlo estimator. Numerical examples are considered for European and barrier call options. The importance sampling estimator for the European call option is given in [2]. For the barrier option we calculate the first order asymptotic approximation and derive the corresponding importance sampling estimator. To further improve the efficiency, randomized quasi-Monte Carlo (RQMC) methods are considered. Randomized quasi-Monte Carlo methods and their applications in security pricing can be found in [6].

4 Some Numerical Results

For numerical experiments we use the same set of parameters given in [2], which are obtained as a result of calibrations to market prices. The set of parameter values that are used in our numerical examples are given in Table 1.

Table 1: Parameter values for our numerical experiments

<table>
<thead>
<tr>
<th>S₀</th>
<th>Y₀</th>
<th>Z₀</th>
<th>K</th>
<th>T</th>
<th>r</th>
<th>m_f</th>
<th>m_s</th>
<th>n_f</th>
<th>n_s</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>-1</td>
<td>-1</td>
<td>50</td>
<td>1</td>
<td>1</td>
<td>-0.8</td>
<td>-0.8</td>
<td>0.5</td>
<td>0.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ρ₁</th>
<th>ρ₂</th>
<th>ρ₁₂</th>
<th>Λ_f</th>
<th>Λ_s</th>
<th>f(y, z)</th>
<th>Δt</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>-0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>exp(y + z)</td>
<td>0.005</td>
<td>10000</td>
</tr>
</tbody>
</table>

In Figure 1 we can see that using any one of the importance sampling estimators significantly reduces the accuracy of the estimates. It should also be noted that the results of the zeroth order term importance sampling estimator are as good as the importance sampling estimator based on the first order asymptotic approximation.

In Table 2 European call option prices are estimated under the two-factor stochastic volatility model with given parameters. In the same table comparing factors of improvements in columns 6 and 7 (in parenthesis), we observe that importance sampling estimator using the zeroth order asymptotic approximation term is more efficient than the one using the first order term by a factor of 10 to 740 times. The big difference in computational efficiency is mainly due to the significant difference in computational costs.

Relative errors and mean squared errors of each estimator is calculated with respect to the “exact” values given in Table 2. These exact values are obtained as a result of Monte Carlo simulations with 30 million sample paths generated for each set of parameters.

In summary our results show that computational efficiency of European and barrier options can be increased significantly using the importance sampling and

---

1 Numerical computations are done in MATLAB with an Intel Core Duo 2.00Ghz processor.
randomized quasi-Monte Carlo methods. In particular, importance sampling estimator based on the zeroth order asymptotic approximation formula outperforms the one derived from a first order approximation.

References


Table 2: European Call Option: factors of improvement of the efficiency with respect to the crude Monte Carlo estimator is given in parenthesis (Black-Scholes Price 10.779, efficiency equals computational time \( \times \) relative error)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \delta )</th>
<th>( P^{MC} )</th>
<th>( \tilde{P} )</th>
<th>( \tilde{P}^{IS}(P_{BS(\sigma)}) )</th>
<th>( \tilde{P}^{IS}(P_{BS(\bar{\sigma})}) )</th>
<th>( \tilde{P}^{IS}(\tilde{P}) )</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.01</td>
<td>11.248</td>
<td>11.069</td>
<td>11.132</td>
<td>11.0864</td>
<td>11.095</td>
<td>11.090</td>
</tr>
<tr>
<td>(1.00)</td>
<td></td>
<td>(0.77)</td>
<td></td>
<td>(7.13)</td>
<td></td>
<td>(0.71)</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.05</td>
<td>10.842</td>
<td>11.208</td>
<td>11.077</td>
<td>11.0497</td>
<td>11.026</td>
<td>11.055</td>
</tr>
<tr>
<td>(1.00)</td>
<td></td>
<td>(9.1)</td>
<td></td>
<td>(26.54)</td>
<td></td>
<td>(0.91)</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.1</td>
<td>11.204</td>
<td>11.449</td>
<td>11.101</td>
<td>11.088</td>
<td>11.147</td>
<td>11.089</td>
</tr>
<tr>
<td>(1.00)</td>
<td></td>
<td>(1.68)</td>
<td></td>
<td>(37)</td>
<td></td>
<td>(0.05)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>11.993</td>
<td>12.201</td>
<td>12.125</td>
<td>12.114</td>
<td>12.121</td>
<td>12.151</td>
</tr>
<tr>
<td>(1.00)</td>
<td></td>
<td>(2.30)</td>
<td></td>
<td>(1.69)</td>
<td></td>
<td>(0.13)</td>
<td></td>
</tr>
</tbody>
</table>

